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Secția I

**MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

1995

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1995

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Secția I

MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

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EXPLICIT FORM FOR THE DISCRETE LOGARITHM OVER THE FIELD $GF(p, k)$

BY

GERASIMOS C. MELETIOU

1. Introduction. The present Note addresses the Discrete Logarithm problem [1], [3], [4], [6]. The problem amounts to finding a quick method (efficient algorithm) for the computation of an integer z satisfying the equation

$$(1) \quad a^z = b,$$

for $b \in GF(p, k)$, given a generator a of the multiplicative group of the field $GF(p, k)$. The main practical interest in the problem stems from cryptography [1]...[4], [6].

In the case that a and z are known the computation of b can be done rapidly (Discrete Exponential Function [4], [7, p. 399]). However, computing z from a and b , that is, computing logarithms over $GF(p, k)$, does not appear to admit a fast algorithm [1], [3], [4].

The integer z in (1) is computed modulo $p^k - 1$. In the case $k=1$ a and b can be regarded as integers from $\{1, 2, \dots, p-1\}$ and z as an integer from $\{1, \dots, p-2\}$. The following polynomial formula has been found [6]

$$(2) \quad z \equiv \sum_{i=1}^{p-2} (1-a^i)^{-1} b^i \pmod{p}.$$

The mere existence of such a formula in terms of powers of b is due to the fact that in a finite field every function from the field to itself can be expressed as a polynomial. Although (2) is of no computational use, still, it is of mathematical interest.

The generalization of (2) to the field $GF(p, k)$, $k > 1$ is the purpose of this correspondence. Integer z in (1) is going to be computed modulo $q-1$, $q=p^k$. Therefore it can be assumed $1 \leq z \leq q-1$ and

$$(3) \quad z = \sum_{m=0}^{k-1} d_m p^m,$$

where $0 \leq d_m \leq p-1$. We use the numeric system with p as a basis, the d_m 's are p -digits. In the case $p=2$ the d_m 's are binary digits (that is bits). For $k=1$ one has $z=d_0$.

It remains to find explicit formulas for the d_m 's. Since $0 \leq d_m \leq q-1$, d_m can be regarded as an element of $GF(p, k)$. The d_m 's are uniquely determined in (3); they are functions of b provided that a is a generator of the multiplicative group of $GF(p, k)$. Then

$$(4) \quad d_m = \sum_{i=1}^{q-2} \frac{b^i}{(1-a^i p^m)}, \quad m=0, 1, \dots, k-1.$$

Trivially, (4) is a generalization of (2). For $p=2$ (4) becomes

$$(5) \quad d_m = \sum \frac{b^i}{(1+a^i)^{2^m}}, \quad m=0, 1, \dots, k-1, \quad d_i \in \{0, 1\}.$$

Therefore in any finite field the discrete logarithm function can be expressed with k polynomials with $q-2$ different coefficients. Surprisingly enough, the formulas for the coefficients are very simple.

2. Main Calculations. Equation (4) has to be shown. For $m=0$ it becomes

$$(6) \quad d_0 = \sum_{i=1}^{q-2} b^i (1-a^i)^{-1}.$$

For the proof Lagrangian Interpolation is going to be used. According to (3) d_0 is the rightmost p -digit of z , thus $d_0 \equiv z \pmod{p}$. The characteristic of the field is p . It follows

$$(7) \quad d_0 = 1 \delta(b, a) + 2 \delta(b, a^2) + \dots + (q-1) \delta(b, a^{q-1}),$$

where $\delta(b, a^j)$ is defined as

$$\delta(b, a^j) = \begin{cases} 1 & b = a^j, \\ 0 & b \neq a^j. \end{cases}$$

Further

$$(8) \quad \delta(b, a^j) = 1 - (b - a^j)^{q-1} = 1 - \sum_{i=0}^{q-1} b^i (-a^j)^{q-1-i} \begin{bmatrix} q-1 \\ i \end{bmatrix}.$$

However, since $q=p^k$ one concludes

$$(9) \quad \begin{bmatrix} q-1 \\ i \end{bmatrix} = \frac{(p^k-1)(p^k-2)\dots(p^k-i)}{i!} \equiv (-1)^i \pmod{p}.$$

In the case $p \neq 2$ the value of $(-1)^{q-1}$ is 1. In the case $p=2$, $(-1)^{q-1} = -1 \equiv 1 \pmod{2}$. Thus (8) implies

$$\delta(b, a^j) = - \sum_{i=1}^{q-1} b^i a^{-ij}.$$

Therefore

$$(10) \quad d_0 = \sum_{j=1}^{q-1} j \left[- \sum_{i=1}^{q-1} b^i a^{-ij} \right] = \sum_{i=1}^{q-1} b^i \left[- \sum_{j=1}^{q-1} i a^{-ij} \right].$$

The sum $-\sum_{j=1}^{q-1} j a^{-ij}$ becomes 0 for $i=q-1$, since $a^{q-1} = 1$, and it becomes $-a^{-i}/(1-a^{-i}) = (1-a^{-i})^{-1}$ in the case $i \neq q-1$. Equality (6) is therefore true.

The above proof for (6) is similar to the proof given by Wells in [6, p. 846] generalized to the field $GF(p, k)$. It becomes clear because of the observation at the end of [6] which states that in the field with $q=p^k$ elements the matrix $M(a) = (a^{ij})$, $0 \leq i, j \leq q-2$ satisfies $M(a)^{-1} = -M(a^{-1})$. Also it is a good idea to be mentioned that $M(a)$ is a discrete Fourier transform over $GF(p, k)$ [5].

It suffices formulas for the d_m 's to be derived, $1 \leq s \leq k-1$. Since $z p^k \equiv z \pmod{q-1}$ it is true

$$(11) \quad a^{z p^k} = b \quad \text{or} \quad (a^{p^s})^{p^{k-s} z} = b.$$

The transformation $x \rightarrow x^{p^s}$ is an automorphism of the field. Therefore a^{p^s} is a generator of the multiplicative group.

According to (3) $p^{k-s} z$ equals to

$$\sum_{m=0}^{s-1} d_m p^{k+m-s} + \sum_{m=s}^{k-1} d_m p^{k+m-s}.$$

The powers of p are for $m \geq s$

$$(12) \quad p^{k+m-s} = p^k p^{m-s} = p^{m-s} \pmod{q-1},$$

therefore

$$(13) \quad p^{k-s} z \equiv v \pmod{q-1},$$

where

$$(14) \quad v = \sum_{m=s}^{k-1} d_m p^{m-s} + \sum_{m=0}^{s-1} d_m p^{k+m-s}.$$

It follows from (14) that $0 \leq v \leq q-1$. Eq. (14) is just the representation of v with p -ary digits. The rightmost p -digit is the coefficient of p^0 that is d_s .

Eq. (11) can be written as

$$(15) \quad (a^{p^s})^v = b.$$

The integer v is the discrete logarithm of b to the basis a^{p^s} . From (6) it is concluded

$$(16) \quad d_s = \sum b^i (1 - a^{p^s i})^{-1} = \sum b^i (1 - a^i)^{-p^s}.$$

The last equation in (16) is true since $x \rightarrow x^{p^s}$ is a field automorphism. The proof is complete.

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FORMĂ EXPLICITĂ PENTRU LOGARITMUL DISCRET PESTE CÂMPUL $GF(p, k)$

(Rezumat)

Pentru un generator al grupului multiplicativ al câmpului $GF(p, k)$, logaritmul discret al unui element b al câmpului de bază a , $b \neq 0$ este întregul z : $1 \leq z \leq p^k - 1$, $b = a^z$. Numerele p -are care îl reprezintă pe z pot fi descrise prin forme polinomiale extrem de simple.

PROPERTIES OF MEASURES OF REPRESENTATION ON A STANDARD H -CONE

BY

LILIANA POPA

0. Introduction. Let S be standard H cone of functions on a semi-saturated set X . We shall use the definitions and notations of [1].

We establish some properties of a measure of representation on a standard H -cone. In [5] we have defined a measure of representation and we have shown that it satisfies the following inequality

$$(1) \quad \sum_{k=0}^n (-1)^{k+1} C_n^k f^{n-k} \sigma(f^k) \geq 0$$

for any f from the algebra of locally bounded functions on a standard H -cone [6]. This inequality has some consequences which were proved in an harmonic space by M e d a and also in a standard H -cone which satisfies the D axiom in [5].

If $X = \mathbb{R}^n$, the correspondence

$$f \mapsto -\Delta f = \sigma(f),$$

where Δ is considered in the sense of distributions, is a measure of representation, for which the harmonic functions are exactly those functions which satisfy

$$\sigma(f) = 0.$$

This give us the idea to define harmonic elements relative to a general measure of representation. We study such elements in the second part of the paper.

If $G \subset X$ is a fine open set, we denote by $S(G)$ the set of all positive functions f on G which are finite on a fine dense subset of G and such that there exists a sequence (s_n) in S , $s_n < \infty$ such that

$$s_n - B^{X \setminus G} s_n \uparrow f,$$

where $B^{X \setminus G} s_n$ is the balayage of s_n on $X \setminus G$. We know that $S(G)$ is a standard H -cone of functions on G [2], for which the fine topology on G given by $S(G)$ coincides with the induced fine topology of X on G .

If G is natural open, then a similar assertion takes place for the natural

topology.

We suppose that the map

$$G \rightarrow S(G), \quad G \subseteq X, \text{ naturally open}$$

is a sheaf, that means the H -cone satisfies natural *sheaf property*.

We suppose that X is *semi-saturated*, that means that any H -integral on S , dominated by an H -measure on X (with respect to S) is also a H -measure on X .

From [2] we know that if X is semi-saturated any fine open subset G of X is semi-saturated with respect to the standard H -cone $S(G)$ of functions on G .

We denote by

$$S_M = \{s \in S \mid \exists \alpha \in \mathbb{R}, s \leq \alpha\}$$

and let $[S_M] = S_M - S_M$. We introduce the next set of functions

$$R = \{f: X \rightarrow \mathbb{R} \mid \exists U_i, i \in I, \text{ natural open such that } f|_{U_i} \in [S_M(U_i)]\}.$$

We have proved in [5] that R is an algebra with the usual product of functions, which is also a lattice for the order relation.

1. Properties of Measures of Representation. Definition. The family of applications $\sigma = (\sigma_U)_{U \subset X, U \text{ naturally open}}$

$$\sigma_U: [S(U)] \rightarrow M(U),$$

where $M(U)$ is the set of all σ -finite measures on U , is called a measure of representation, if:

1. σ_U is linear;
2. $\forall f, g \in S(U)$, with $f \leq g$, the next equivalence is valid:

$$fg \Leftrightarrow \sigma_U(f) \leq \sigma_U(g);$$

3. for any U_1, U_2 naturally open, with $U_1 \subset U_2$ and $f \in S(U_2)$,

$$\sigma_{U_1}(f|_{U_1}) = \sigma_{U_2}(f)|_{U_1}.$$

If σ is a measure of representation and h is a function on X , $h > 0$, then the application

$$\sigma^h(f) = h\sigma(f)$$

defines another measure of representation.

For any $h \in R$, $h > 0$, we define

$$R^{(h)} = \{f/h \mid f \in R\};$$

from [7] we know that $1/h \in R$, hence $R^{(h)}$ is an algebra. The map

$$\sigma^h(f) = \sigma(fh)$$

is a measure of representation. In [5] we have constructed an example of a measure of representation on an autodual H -cone, which satisfies the axiom D. If we drop the axiom D, in [6], we have proved that the inequality (1) is valid too.

The next Proposition allows us to define a notion of gradient measure, which is positive.

Proposition 1.1. For any $f, g, h \in R$, the next equality is valid:

$$\sigma(fgh) - f\sigma(g)h - g\sigma(fh) - h\sigma(fg) + fg\sigma(h) + fh\sigma(g) + gh\sigma(f) - fgh\sigma(1) = 0.$$

P r o o f. For $n=3$ we get from (1)

$$\sigma(f^3) - 3f\sigma(f^2) + 3f^2\sigma f - f^3\sigma(1) \geq 0,$$

for any $f \in R$. Applying this to $(-f)$ we obtain the converse inequality, hence

$$(2) \quad \sigma(f^3) - 3f\sigma(f^2) + 3f^2\sigma(f) - f^3\sigma(1) = 0,$$

for any $f \in R$.

If $f, g, h \in R$, we consider the function $f = f + tg + sh$, $t, s \in R$; from the previous equality, we get that the coefficient of ts annihilates, and therefore we obtain (2).

If in (1) we take $n=2$ we get

C o r o l l a r y.

$$2f\sigma(f) - \sigma(f^2) - f^2\sigma(1) \geq 0.$$

Using Proposition 1, we get that for any $n \geq 3$

$$\sum_{k=0}^n (-1)^{k+1} C_n^k f^{n-k} \sigma(f^k) = 0.$$

P r o p o s i t i o n 1.2. For any $f, g \in R$, we have

$$\sum_{k=0}^{n-1} (-1)^k C_{n-1}^k f^{(n-1)-k} \sigma(f^k g) = g \sum_{k=0}^{n-1} C_{n-1}^k f^{(n-1)-k} \sigma(f^k).$$

2. Harmonic and Potential Elements Relative to a Measure of Representation. Let σ be a measure of representation.

We define the sets

$$H_\sigma = \{h \in S \mid \sigma(h) = 0\},$$

$$P_\sigma = \{p \in S \mid p \wedge h = 0, \forall h \in H_\sigma\}.$$

D e f i n i t i o n. The elements of H_σ , respectively P_σ are called *harmonic*, respectively *potentials* relative to the measure of representation σ .

It is easy to see that H_σ , P_σ are bands of S with respect to the specific order; that means that:

1. H_σ and P_σ are convex subcones of S .
2. H_σ and P_σ are specifically solid.
3. For any family $(s_i) \in H_\sigma$, $i \in I$, respectively P_σ for which $\sum_{i \in I} s_i \in S$,

$$\sum_{i \in I} s_i \in H_\sigma, \text{ respectively } \sum_{i \in I} s_i \in P_\sigma.$$

P r o p o s i t i o n 2.1.1. If $h \in S$ satisfies $\sigma(h) = 0$, then h is substractible.

2. If $\sigma(1) = 0$, then every $h \in S_H$, which is bounded, satisfies $\sigma(h) = 0$.

P r o o f. Indeed, if $h \leq s$, and $\sigma(h) = 0 \leq \sigma(s)$, then from the Axiom 2, it results $h \prec s$. The second assertion is immediate.

P r o p o s i t i o n 2.2 Any $s \in S$ possesses a decomposition of the form

$$s = h + p, \quad h \in H_\sigma, \quad p \in P_\sigma,$$

where h, p are uniquely determined. If $p \in P_\sigma$ and $s \in S$ is such that $s \leq p$, then $s \in P_\sigma$.

Proof. For $s \in S$, let,

$$h = \bigvee \{s' \mid s' \in H_\sigma, s' \leq s\}.$$

Obviously, $\sigma(h) = 0$, hence $h \in H_\sigma$. We define $p = s - h$. Let $u = (s - h) \wedge h'$, where $h' \in H_\sigma$. Because $u \leq h'$, it is from H_σ and $u \leq s - h$. Therefore $s = u + h + h''$ and $h + h''$ is from H_σ and greater than h . Hence $u = 0$.

If we suppose that $s = h' + p'$, with $h' \in H_\sigma, p' \in P_\sigma$, it follows that $h + p = h' + p'$. But $h \leq h'$ and therefore from $h - h' = p' - p$ we deduce that $p = p'$ and $h = h'$.

Let now $p \in P_\sigma$ and $s \in S, s \leq p$. If we write

$$s = h' + p', \quad h' \in S_\sigma, \quad p' \in P_\sigma,$$

then $h' \leq p, h' \leq p$, hence $h = 0$. It follows that

$$s = p'.$$

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PROPRIETĂȚI ALE MĂSURILOR DE REPREZENTARE ÎNTR-UN H-CON STANDARD

(Rezumat)

Se extind unele din proprietățile măsurilor de reprezentare la cazul mai general, al unui H -con standard, care nu satisface axioma D. Se obțin astfel generalizări ale rezultatelor din [4] și [5].

Se definește apoi noțiunea de element armonic și potențial, relativ la o măsură de reprezentare, și se dă o teoremă de descompunere pentru orice $s \in S$ în funcție de aceste elemente.

SOUS-VARIÉTÉS CAUCHY-RIEMANN DES VARIÉTÉS PRESQUE COSYMPLECTIQUES À FEUILLES KÄHLER

PAR

LANCINE FOFANA

0. Introduction. Les variétés presque cosymplectiques ont été introduites par D. B l a i r [4] et étudiées par certains auteurs (voir bibliographie). La notion de variété presque cosymplectique à feuilles Kähler a été introduite par Z. O l s z a k [7]. Le feuilletage canonique des variétés presque cosymplectiques a été considéré dans [5]. Le concept de sous-variété Cauchy-Riemann a été initié en 1978 par A. B e j a n c u. L'objectif de ce présent ouvrage est l'étude de l'intégrabilité des distributions canoniques d'une sous-variété Cauchy-Riemann d'une variété presque cosymplectique à feuilles Kähler.

1. Préliminaires. Variété cosymplectique à feuilles Kähler. Soit $\bar{M}$ une variété presque de contact métrique à structure (φ, ξ, η, g) , où φ est un champ de tenseurs de type (1,1), ξ est un champ de vecteurs, η est une 1-forme et g est la métrique Riemannienne sur $\bar{M}$ qui satisfont aux relations suivantes:

$$(1.1) \quad \varphi \xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1;$$

$$(1.2) \quad \varphi^2 X = -X + \eta(X)\xi, \quad g(X, \xi) = \eta(X);$$

$$(1.3) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

pour tous champs de vecteurs X, Y tangents à $\bar{M}$.

Dans ce qui suit une telle variété sera notée simplement par $\bar{M}^{2n+1}(\varphi, \xi, \eta, g)$.

La variété $\bar{M}^{2n+1}(\varphi, \xi, \eta, g)$ est appelée variété presque cosymplectique si $d\eta = 0$ et $d\Phi = 0$ où $\Phi(X, Y) = g(\varphi X, Y)$, pour tous X, Y tangents à $\bar{M}$.

Sur la variété presque cosymplectique $\bar{M}$ considérons le champ de tenseurs Λ de type (1,1), défini par

$$(1.4) \quad \Lambda X = -\bar{\nabla}_X \xi,$$

où $\bar{\nabla}$ est la connexion de Levi-Civita sur $\bar{M}$, et le champ de tenseurs K de type (0,2)

défini par

$$(1.5) \quad K(X, Y) = g(\Lambda X, Y)$$

pour tous X, Y tangents à $\bar{M}$.

Le champ de tenseurs K est symétrique; en plus on a

$$(1.6) \quad \Lambda\varphi + \varphi\Lambda = 0, \quad \Lambda\xi = 0.$$

Soit $\mathcal{D}$ une distribution sur $\bar{M}$ définie par $\eta = 0$, et $\mathcal{F}$ un feuilletage de codimension un sur $\bar{M}$ correspondant à $\mathcal{D}$ et $\bar{M}$, une feuille de $\mathcal{F}$. Le champ de vecteurs ξ est toujours orthogonale à $\bar{M}$; puisque $\varphi\mathcal{D} = \mathcal{D}$, l'opérateur linéaire φ est invariant sur $\bar{M}$. Donc φ induit un champ de tenseurs sur $\bar{M}$ par $J\tilde{X} = \varphi\tilde{X}$, pour tout $\tilde{X}$ tangent à $\bar{M}$. Soit G la métrique Riemannienne induite sur $\bar{M}$ par $G(\tilde{X}, \tilde{Y}) = g(\tilde{X}, \tilde{Y})$ pour tous $\tilde{X}, \tilde{Y}$ tangents à $\bar{M}$. Évidemment $d\Phi = 0$ et (J, G) est une structure presque Kähler. Ainsi $\bar{M}$ est une variété presque cosymplectique à feuilles Kähler.

Proposition. Soit $\bar{M}$ une variété presque cosymplectique à structure presque cosymplectique (φ, ξ, η, g) .

$\bar{M}$ est une variété presque cosymplectique à feuilles Kähler si et seulement si (φ, ξ, η, g) vérifie

$$(1.7) \quad (\bar{\nabla}_X \varphi)Y = -g(\varphi\Lambda X, Y)\xi + \eta(Y)\varphi\Lambda X.$$

Corollaire. Une variété presque cosymplectique à feuilles Kähler est une variété cosymplectique si et seulement si

$$(1.8) \quad \bar{\nabla}_X \xi = 0.$$

Une variété presque de contact métrique $\bar{M}^{2n-1}(\varphi, \xi, \eta, g)$ est une variété presque cosymplectique à feuilles Kähler $\Leftrightarrow \exists \Lambda$ un champ de tenseurs de type $(1,1)$ défini par $\Lambda X = -\bar{\nabla}_X \xi$ vérifiant les relations suivantes:

i) Λ est symétrique, c'est-à-dire $g(\Lambda X, Y) = g(\Lambda Y, X)$;

ii) $\Lambda\xi = 0, \Lambda\varphi + \varphi\Lambda = 0$;

iii) $(\bar{\nabla}_X \varphi)Y = -g(\varphi\Lambda X, Y)\xi + \eta(Y)\varphi\Lambda X$.

2. Sous-variété Cauchy-Riemann. Soit M une variété Riemannienne de dimension m , isométriquement immergée dans $\bar{M}$ et supposons que le champ de vecteurs ξ est tangent à M . Soit $\{\xi\}$, la distribution engendrée par ξ . Notons respectivement par TM et $T^\perp M$, le fibré tangent de M et le fibré normal de M , de même notons par $T_x M$ et $T_x^\perp M$ l'espace vectoriel des vecteurs tangents en x à M respectivement l'espace vectoriel des vecteurs normaux (covecteurs) en x à M .

Définition. La sous-variété M d'une variété presque cosymplectique à feuilles Kähler est appelée sous-variété Cauchy-Riemann (ou simplement CR sous-variété) si M est dotée d'une paire de distributions différentiables orthogonales $(D, D^\perp)$ que vérifient les relations suivantes:

i) $TM = D \oplus D^\perp \oplus \{\xi\}$.

ii) La distribution D est invariante par rapport à φ c'est-à-dire $\varphi D_x = D_x$ pour

tout x de M .

iii) La distribution $D^\perp$ est anti-invariante par rapport à φ , c'est-à-dire $\varphi D_x^\perp \subset T_x^\perp M$, pour tout x de M .

Les distributions D et $D^\perp$ sont appelées distribution invariante (ou distribution horizontale) respectivement distribution anti-invariante (ou distribution verticale).

Une CR sous-variété est appelée invariante (resp. anti-invariante) si $D_x^\perp = \{0\}$ (resp. $D_x = \{0\}$) pour tout x de M . Une CR sous-variété qui n'est ni invariante ni anti-invariante est appelée propre.

Soit ∇ (resp. $\nabla^\perp$) la connexion de Levi-Civita sur M (resp. la connexion linéaire induite de $\bar{\nabla}$ sur $T^\perp M$); alors les formules de Gauss et Weingarten deviennent respectivement:

$$(2.1) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \text{ pour tous } X, Y \text{ de } TM,$$

$$(2.2) \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \text{ pour tout } X \text{ de } TM \text{ et } N \text{ de } T^\perp M,$$

où h est la deuxième forme fondamentale de M et A est l'opérateur de Weingarten; h et A étant liés par la relation suivante

$$(2.3) \quad g(h(XY), N) = g(A_N X, Y) \text{ pour tous } X, Y \text{ de } TM \text{ et } N \text{ de } T^\perp M.$$

Pour tout champ de vecteurs X tangent à M , on pose

$$(2.4) \quad X = PX + QX + \eta(X)\xi,$$

où PX et QX appartiennent respectivement à D et $D^\perp$.

Pour tout champ de vecteurs N normal à M on pose

$$(2.5) \quad \varphi N = BN + CN,$$

où BN (resp. CN) représente la composante tangentielle (resp. composante normale) de φN .

Proposition 2.1. Soit M une sous-variété Cauchy-Riemann d'une variété presque cosymplectique à feuilles Kähler M . Alors les relations suivantes ont lieu:

$$(2.6) \quad P\nabla_X \varphi PY - PA_{\varphi QY} X = \varphi P\nabla_X Y - \eta(Y) \varphi P\nabla_X \xi,$$

$$(2.7) \quad Q\nabla_X \varphi PY - QA_{\varphi QY} X = Bh(X, Y) - \eta(Y) Bh(X, \xi),$$

$$(2.8) \quad \eta(\nabla_X \varphi PY - QA_{\varphi QY} X) = g(\varphi \nabla_X \xi, Y) + g(\varphi h(X, \xi), Y),$$

$$(2.9) \quad h(X, \varphi PY) + \nabla_X^\perp \varphi QY = Ch(X, Y) + \varphi Q\nabla_X Y - \eta(Y) Ch(X, \xi),$$

pour tous X, Y de TM .

Démonstration. Pour tous X, Y de TM , en tenant compte des relations (2.1), (2.2), (2.4) et (2.5), nous avons d'une part

$$\begin{aligned}
(\bar{\nabla}_X \varphi)Y &= \bar{\nabla}_X \varphi Y - \varphi \bar{\nabla}_X Y = \bar{\nabla}_X (\varphi P Y + \varphi Q Y) - \varphi (\nabla_X Y + h(X, Y)) = \\
&= \nabla_X \varphi P Y + h(X, \varphi P Y) - A_{\varphi Q Y} X + \nabla_X^\perp \varphi Q Y - \varphi \nabla_X Y - \varphi h(X, Y) = \\
&= P \nabla_X \varphi P Y + Q \nabla_X \varphi P Y + \eta (\nabla_X \varphi P Y) \xi + h(X, \varphi P Y) - P A_{\varphi Q Y} X - \\
&\quad - Q A_{\varphi Q Y} X - \eta (A_{\varphi Q Y} X) \xi + \nabla_X^\perp \varphi Q Y - \varphi P \nabla_X Y - \varphi Q \nabla_X Y - B h(X, Y) - C h(X, Y),
\end{aligned}$$

et d'autre part

$$\begin{aligned}
(\bar{\nabla}_X \varphi)Y &= g(\varphi \bar{\nabla}_X \xi, Y) \xi - \eta(Y) \varphi \bar{\nabla}_X \xi = \\
&= g(\varphi \nabla_X \xi, Y) \xi + g(\varphi h(X, \xi), Y) \xi - \eta(Y) \varphi \nabla_X \xi - \eta(Y) \varphi h(X, \xi).
\end{aligned}$$

De la décomposition (2.4) résultent aisément les relations de la proposition (2.1).

Proposition 2.2. Soit M une sous-variété Cauchy-Riemann d'une variété presque cosymplectique à feuilles Kähler $\bar{M}$. Alors les relations suivantes ont lieu:

$$(2.10) \quad P \nabla_X B N - P A_{CN} X = \varphi P A_N X,$$

$$(2.11) \quad Q \nabla_X B N - Q A_{CN} X = B \nabla_X^\perp N,$$

$$(2.12) \quad \eta(\nabla_X B N - A_{CN} X) = -g(\varphi \Lambda X, N),$$

$$(2.13) \quad h(X, B N) + \nabla_X^\perp C N + \varphi Q A_N X = C \nabla_X^\perp N,$$

pour tout X de TM et N de $T^\perp M$.

Démonstration. En tenant compte des relations (2.1), (2.2), (2.4) et (2.5), nous avons d'une part

$$\begin{aligned}
(a) \quad (\bar{\nabla}_X \varphi)N &= \bar{\nabla}_X \varphi N - \varphi \bar{\nabla}_X N = \bar{\nabla}_X (B N + C N) + \varphi A_N X - \varphi \nabla_X^\perp N = \nabla_X B N + \\
&\quad + h(X, B N) - A_{CN} X + \nabla_X^\perp C N + \varphi P A_N X = \varphi Q A_N X - B \nabla_X^\perp N - C \nabla_X^\perp N,
\end{aligned}$$

et d'autre part

$$(b) \quad (\bar{\nabla}_X \varphi)N = -g(\varphi \Lambda X, N) \xi.$$

De (a) et (b) résultent aisément les affirmations de la proposition (2.2).

Proposition 2.3. Soit M une sous-variété Cauchy-Riemann d'une variété presque cosymplectique à feuilles Kähler $\bar{M}$. Alors pour tous X, Y de $D^\perp$ on a

$$(2.14) \quad A_{\varphi Y}^X = A_{\varphi X}^Y.$$

Démonstration. Pour tout champ de vecteurs Z tangent à M , en tenant

compte de (2.3), (2.1) et (1.3), on a d'une part

$$\begin{aligned} g(A_{\varphi X}^Y Z) &= g(h(Z, Y), \varphi X) = g(\bar{\nabla}_X Y - \nabla_X Y - \nabla_Z Y \varphi X) = \\ &= g(\bar{\nabla}_Z Y, \varphi X) = -g(\varphi \nabla_Z Y, X). \end{aligned}$$

D'autre part on a $(\bar{\nabla}_Z \varphi)Y = -g(\varphi \wedge Z, Y)\xi$, ou encore $\varphi \bar{\nabla}_Z Y = \bar{\nabla}_Z \varphi Y + g(\varphi \wedge Z, Y)\xi$, d'où

$$\begin{aligned} g(A_{\varphi X}^Y Z) &= -g(\nabla_Z \varphi Y + g(\varphi \wedge Z, Y)\xi, X) = \\ &= -g(\bar{\nabla}_Z \varphi Y, X) - g(\varphi \wedge Z, Y)g(\xi, X) = -g(\bar{\nabla}_Z \varphi Y, X) = g(A_{\varphi Y}^X Z). \end{aligned}$$

Proposition 2.4. Soit M une sous-variété Cauchy-Riemann d'une variété cosymplectique à feuilles Kähler $\bar{M}$. Alors pour tous X, Y de $D^\perp$ on a

$$(2.15) \quad [X, Y] \in D \oplus D^\perp.$$

Démonstration:

$$\begin{aligned} g([X, Y], \xi) &= g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, \xi) = g(\bar{\nabla}_X Y, \xi) - g(\bar{\nabla}_Y X, \xi) = \\ &= -g(Y, \bar{\nabla}_X \xi) + g(X, \bar{\nabla}_Y \xi) = g(Y, \wedge X) - g(X, \wedge Y) = 0. \end{aligned}$$

Donc $[X, Y] \in D \oplus D^\perp$.

3. Intégrabilité des distributions. Théorème 1. Soit M une sous-variété Cauchy-Riemann d'une variété presque cosymplectique à feuilles Kähler $\bar{M}$. Alors la distribution anti-invariante est involutive.

Démonstration. Pour tous X, Y de $D^\perp$, il suffit de montrer que $[X, Y]$ est de $D^\perp$; $(\bar{\nabla}_X \varphi)Y = -g(\varphi \wedge X, Y)\xi = g(\wedge X, \varphi Y)\xi$, ou encore $\bar{\nabla}_X \varphi Y - \varphi \bar{\nabla}_X Y = g(\wedge X, \varphi Y)\xi$ ou encore

$$(3.1) \quad -A_{\varphi Y}^X + \nabla_X^\perp \varphi Y - \varphi \bar{\nabla}_X Y = g(\wedge X, \varphi Y)\xi.$$

En substituant X à Y on a

$$(3.2) \quad -A_{\varphi X}^Y + \nabla_Y^\perp \varphi X - \varphi \bar{\nabla}_Y X = g(\wedge Y, \varphi X)\xi.$$

De (3.1) et (3.2) on a

$$-\varphi[X, Y] + \nabla_X^\perp \varphi X - \nabla_Y^\perp \varphi Y + A_{\varphi X}^Y - A_{\varphi Y}^X = g(\wedge X, \varphi Y)\xi - g(\wedge Y, \varphi X)\xi.$$

En tenant compte de (2.14) on a

$$(3.3) \quad -\varphi[X, Y] = \nabla_Y^\perp \varphi X - \nabla_X^\perp \varphi Y + g(\wedge X, \varphi Y)\xi - g(\wedge Y, \varphi X)\xi.$$

En appliquant φ à (3.3) en tenant compte de (2.15) on a

$$[X, Y] = \varphi(\nabla_Y^\perp \varphi X - \nabla_X^\perp \varphi Y).$$

Mais puisque $\nabla_Y^\perp \varphi X - \nabla_X^\perp \varphi Y$ est de $\varphi D^\perp$, alors $[X, Y]$ est de $D^\perp$.

Théorème 2. Soit M une sous-variété Cauchy-Riemann d'une variété

presque cosymplectique à feuilles Kähler $\bar{M}$. Alors la distribution $D \oplus D^\perp$ est involutive.

Démonstration. Pour tous X, Y de TM nous avons

$$\begin{aligned} g([X, Y], \xi) &= g(\bar{\nabla}_X Y, \xi) - g(\bar{\nabla}_Y X, \xi) = -g(Y, \bar{\nabla}_X \xi) + g(X, \bar{\nabla}_Y \xi) = \\ &= g(Y, \wedge X) - g(X, \wedge Y) = 0. \end{aligned}$$

D'où $[X, Y] \in D \oplus D^\perp$.

Théorème 3. *Soit M une sous-variété Cauchy-Riemann d'une variété presque cosymplectique à feuilles Kähler $\bar{M}$. Alors la distribution $D \oplus \{\xi\}$ est involutive si et seulement si la deuxième forme fondamentale de M satisfait à la relation suivante:*

$$h(X, \varphi Y) = h(\varphi X, Y),$$

pour tous X, Y de D .

Démonstration. De (2.9) nous avons

$$h(X, \varphi Y) = ch(X, Y) + \varphi Q \nabla_X Y,$$

d'où $h(X, \varphi Y) - h(Y, \varphi X) = \varphi Q[X, Y]$.

Donc $[X, Y] \in D \oplus \{\xi\}$ si et seulement si $h(X, \varphi Y) = h(Y, \varphi X)$.

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CR-SUBVARIETĂȚI ALE VARIETĂȚILOR APROAPE COSIMPLECTICE CU FOLIATE KÄHLER

(Rezumat)

Se studiază integrabilitatea distribuțiilor canonice ale unei CR-subvarietăți a unei varietăți aproape cosimplete cu foliate Kähler.

A GENERALIZATION OF SCOINS-GLICKSMAN'S THEOREM

BY

EMILIA POPOVICI

Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of vertices and $U = \{u_1, u_2, \dots, u_q\}$ a set of $q = \binom{n}{2}$ edges joining pairs of vertices from X . For a graph $G = (X, V)$ with n vertices and p connected components which contain respectively $n_1, n_2, \dots, n_p$ vertices, we denote $m(V) = n_1 n_2 \dots n_p$ if each of the all p connected components of the graph G is a tree and, otherwise $m(V) = 0$, (i.e. if the graph G contains at least a cycle).

By $T(X, U)$ we denote the number of the trees which can be constructed using the vertices from X and with those edges which do not belong to the set U .

Theorem (Austin). *If U is a set of edges joining the vertices from the set $S \subset U$, with $|S| = s$, in all possible ways, (i.e. (S, U) is a complete graph), then*

$$(1) \quad T(X, U) = n^{n-2} \left[1 - \frac{s}{n} \right]^{s-1}.$$

Theorem (Scoins, Glicksman). *If the graph (X, U) is a reunion of two disjunctive complete graphs, (S, V) and (T, W) , i.e. $S \cup T = X$, $S \cap T = \emptyset$, and $V \cup W = U$, with $|S| = s$, $|T| = t$, then the number of the trees having X as a set of vertices and which do not contain any edge from U is*

$$(2) \quad T(X, U) = s^{t-1} t^{s-1}.$$

Generalization. *If the graph (X, U) is the reunion of r complete graphs which are disjunctive two by two ($r \geq 2$), $(X_1, U_1), (X_2, U_2), \dots, (X_r, U_r)$, i.e. $X = \bigcup_{i=1}^r X_i$, $X_i \cap X_j = \emptyset$ for $i \neq j$ and $i, j = \overline{1, r}$, $U = \bigcup_{i=1}^r U_i$, with $|X_i| = s_i$, $i = \overline{1, r}$, then the number of the trees having X as a set of vertices and which do not contain any edge from U is*

$$(3) \quad T(X, U) = n^{r-2} (n-s_1)^{s_1-1} (n-s_2)^{s_2-1} \dots (n-s_r)^{s_r-1}.$$

P r o o f. We shall show that the following relation holds:

$$(4) \quad \frac{T(X, U_1 \cup U_2 \cup \dots \cup U_r)}{n^{n-2}} = \frac{T(X, U_1)}{n^{n-2}} \frac{T(X, U_2)}{n^{n-2}} \dots \frac{T(X, U_r)}{n^{n-2}}.$$

According to T e m p e r l e y's theorem, the number of the trees constructed by using the vertices of the set X with n elements and whose edges do not belong to the set U , is

$$(5) \quad T(X, U) = n^{n-2} \sum_{V \subset U} m(V) \left[-\frac{1}{n} \right]^{|V|}.$$

Then the equality (4) is equivalent to

$$(6) \quad \sum_{Z \subset U_1 \cup U_2 \dots \cup U_r} m(Z) \left[-\frac{1}{n} \right]^{|Z|} = \sum_{Z_1 \subset U_1} m(Z_1) \left[-\frac{1}{n} \right]^{|Z_1|} \sum_{Z_2 \subset U_2} m(Z_2) \left[-\frac{1}{n} \right]^{|Z_2|} \dots \sum_{Z_r \subset U_r} m(Z_r) \left[-\frac{1}{n} \right]^{|Z_r|}.$$

If $Z \subset U_1 \cup U_2 \dots \cup U_r$, and the r complete graphs are disjunctive two by two, ge then $Z = Z_1 \cup \dots \cup Z_r$, with $Z_i \subset U_i$, $i = \overline{1, r}$ and relation (6) becomes

$$(7) \quad m(Z) \left[-\frac{1}{n} \right]^{|Z|} = m(Z_1) \left[-\frac{1}{n} \right]^{|Z_1|} \dots m(Z_r) \left[-\frac{1}{n} \right]^{|Z_r|}.$$

We have the following cases:

a) If there exists at least a graph (X, Z_i) which contains cycles, i.e. $m(Z_i) = 0$, then (X, Z) will also contain cycles, i.e. $m(Z) = 0$. In this case, the both members of relation (7) are zero.

b) If $Z_i \neq \emptyset$, for any $i = \overline{1, r}$ and if (X, Z_1) has p_1 connected components which are trees with $n_{11}, n_{12}, \dots, n_{1p_1}$ vertices respectively; (X, Z_2) has p_2 connected components which are trees with $n_{21}, n_{22}, \dots, n_{2p_2}$ vertices, respectively...; (X, Z_r) has p_r connected components which are trees with $n_{r1}, n_{r2}, \dots, n_{rp_r}$ vertices, respectively, and the sets Z_i are disjunctive two by two, then it follows that (X, Z) has $p_1 + p_2 + \dots + p_r$ connected components which are trees and contain respectively $n_{11}, n_{12}, \dots, n_{1p_1}; n_{21}, n_{22}, \dots, n_{2p_2}; \dots; n_{r1}, n_{r2}, \dots, n_{rp_r}$ vertices. In this case

$$m(Z) = m(Z_1) m(Z_2) \dots m(Z_r) = n_{11} n_{12} n_{1p_1} \dots n_{r1} n_{r2} \dots n_{rp_r}$$

and

$$|Z| = |Z_1| + \dots + |Z_r|;$$

therefore the relation (7) is true.

c) If at least one of the sets Z_i , $i = \overline{1, r}$ is empty, in this case $m(Z_i) = 1$ (because all the components of (X, Z_i) are points. For (X, Z_i) , with $Z_i = \emptyset$, by a similar argument to that from b), the relation (7) immediately follows. If (7) is true, it follows that (4) is also true. From (4) we have

$$T(X, U) = \frac{1}{(n^{n-2})^{r-1}} T(X, U_1) T(X, U_2) \dots T(X, U_r).$$

According to A u s t i n's theorem we get

$$\begin{aligned} T(X, U) &= \frac{1}{(n^{n-2})^{r-1}} n^{n-2} \left[1 - \frac{s_1}{n} \right]^{s_1-1} n^{n-2} \left[1 - \frac{s_2}{n} \right]^{s_2-1} \dots n^{n-2} \left[1 - \frac{s_r}{n} \right]^{s_r-1} = \\ &= n^{n-2} \left[1 - \frac{s_1}{n} \right]^{s_1-1} \left[1 - \frac{s_2}{n} \right]^{s_2-1} \dots \left[1 - \frac{s_r}{n} \right]^{s_r-1} \end{aligned}$$

and consequently

$$T(X, U) = \frac{n^{n-2}}{n^{s_1+s_2+\dots+s_r-r}} (n-s_1)^{s_1-1} (n-s_2)^{s_2-1} \dots (n-s_r)^{s_r-1}.$$

Because $n = s_1 + s_2 + \dots + s_r$ we obtain

$$T(X, U) = n^{r-2} (n-s_1)^{s_1-1} (n-s_2)^{s_2-1} \dots (n-s_r)^{s_r-1}.$$

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O GENERALIZARE A TEOREMEI LUI SCOINS ȘI GLICKSMAN

(Rezumat)

Se prezintă o metodă de numărare a arborilor ca aplicație la principiul includerii și al excluderii. Această metodă este o generalizare a teoremei lui Scoins și Glicksman.

LES TYPES DE CIRCULATIONS ADMISSIBLES DANS UN GRAPHE ESCALIER NON ORIENTÉ (I)

PAR

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Soit un graphe escalier [2] (Fig. 1) où les arêtes sont non orientées; les flèches du graphe indiquent le sens de la circulation qui sera introduit plus en bas.

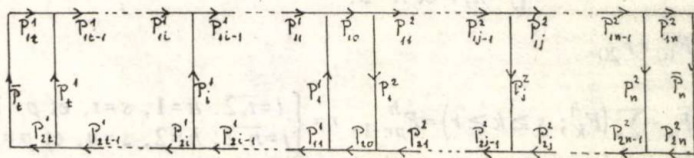


Fig. 1 — G^f avec $\bar{b}/c' \bar{c}'/b$.

Nous noterons les fonctions de circulation et de capacité comme suit:

$$F(P_r^h) = F_r^h, \quad c(P_r^h) = c_r^h, \quad C(P_r^h), \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \quad k = 1, 2, \\ j = \overline{1, n}, & h = 2, \quad k = 1, 2; \end{cases}$$

$$F(P_{kr}^h) = F_{kr}^h, \quad c(P_{kr}^h) = c_{kr}^h, \quad C(P_{kr}^h), \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \quad k = 1, 2, \\ j = \overline{1, n}, & h = 2, \quad k = 1, 2; \end{cases}$$

$$F(\bar{P}_s) = \bar{F}_s, \quad c(\bar{P}_s) = \bar{c}_s, \quad C(\bar{P}_s) = \bar{C}_s, \quad s = t, n;$$

$$F(P_{k0}) = F_{k0}, \quad c(P_{k0}) = c_{k0}, \quad C(P_{k0}) = C_{k0}, \quad k = 1, 2.$$

Définition 1. On appelle une circulation dans un graphe escalier G [2], une paire (f, F) où f est une orientation $\bar{b}/c' \bar{c}'/b$ ou $\bar{b}/c \bar{c}/b$ (équivalentes) [2] de G (Fig. 1) et F est une circulation dans G^f (le graphe orienté attaché), c'est à dire on satisfait les équations de conservation:

$$\bar{F}_s = F_{1s}^h = F_{1s}^h, \quad s = \begin{cases} t & \text{et } h=1, \\ n & \text{et } h=2; \end{cases}$$

$$(1) \quad F_{kr-1}^h = F_{kr}^h + F_r^h, \quad k=1,2; \quad r = \begin{cases} i=\overline{2,t} & \text{et } h=1, \\ j=\overline{2,n} & \text{et } h=2; \end{cases}$$

$$F_{k0} = F_{k1}^h + F_{k1}^h, \quad k=1,2 \text{ et } h=1,2.$$

Cette circulation est admissible, si elle satisfait les conditions:

$$\bar{c}_s \leq \bar{F}_s \leq \bar{C}_s, \quad s=t,n;$$

$$(2) \quad c_r^h \leq F_r^h \leq C_r^h, \quad r = \begin{cases} i=\overline{1,t} & \text{et } h=1, \\ j=\overline{1,n} & \text{et } h=2; \end{cases}$$

$$c_{kr}^h \leq F_{kr}^h \leq C_{kr}^h, \quad r = \begin{cases} i=\overline{1,t} & \text{et } h=1, \\ j=\overline{1,n} & \text{et } h=2; \end{cases} \quad k=1,2;$$

$$c_{k0} \leq F_{k0} \leq C_{k0}, \quad k=1,2.$$

Remarque 1. De (1), on vérifie par induction que

$$F_{1r}^h = F_{2r}^h, \quad r = \begin{cases} i=\overline{1,t} & \text{et } h=1, \\ j=\overline{1,n} & \text{et } h=2; \end{cases}$$

$$F_{10} = F_{20};$$

$$\bar{F}_s + \sum (F_k^h; s \geq k \geq r) = F_{pr-1}^h, \quad r = \begin{cases} i=\overline{1,2}, h=1, s=t, \text{ et } p=1,2, \\ j=\overline{1,2}, h=2, s=n, \text{ et } p=1,2; \end{cases}$$

$$F_{k0} = \bar{F}_s + \sum (F_r^h; s \geq r \geq 1), \quad r = \begin{cases} i=\overline{1,t} & \text{et } h=1, s=t, k=1,2, \\ j=\overline{1,n} & \text{et } h=2, s=n, k=1,2. \end{cases}$$

On définit les quantités:

$$c'_{hs} = \max(c'_{1s}, \bar{c}_s, c'_{2s}) \quad \text{et} \quad C'_{hs} = \min(C'_{1s}, \bar{C}_s, C'_{2s}) \quad \text{pour } h = \begin{cases} 1 & \text{et } s=t, \\ 2 & \text{et } s=n; \end{cases}$$

$$c'_{hr-1} = \max(c'_{1r-1}, c'_{hr} + c_r^h, c'_{2r-1}), \quad r = \begin{cases} i=\overline{1,2} & \text{et } h=1, \\ j=\overline{1,2} & \text{et } h=2; \end{cases}$$

$$C'_{hr-1} = \min(C'_{1r-1}, C'_{hr} + C_r^h, C'_{2r-1}), \quad r = \begin{cases} i=\overline{1,2} & \text{et } h=1, \\ j=\overline{1,2} & \text{et } h=2; \end{cases}$$

$$c'_{10} = c'_{20} = \max(c_{10}, c'_{11} + c_1^1, c'_{21} + c_1^2, c_{20}),$$

$$C'_{10} = C'_{20} = \min(C_{10}, C'_{11} + C_1^1, C'_{21} + C_1^2, C_{20}).$$

Remarque 2. Nous avons:

$$c'_{hr-1} \geq c'_{hr}, \quad r = \begin{cases} i=\overline{t,1} \text{ et } h=1, k=1,2; \\ j=\overline{n,1} \text{ et } h=2, k=1,2; \end{cases}$$

$$\max(c^h_{kr}, c^h_r) \leq c'_{hr}, \quad r = \begin{cases} i=\overline{t,1} \text{ et } h=1, k=1,2; \\ j=\overline{n,1} \text{ et } h=2 \text{ et } k=1,2; \end{cases}$$

$$\min(C^h_{kr}, C^h_r) \geq C'_{hr}, \quad r = \begin{cases} i=\overline{t,1} \text{ et } h=1, k=1,2; \\ j=\overline{n,1} \text{ et } h=2 \text{ et } k=1,2; \end{cases}$$

$$c'_{h0} \geq c'_{hr} + \sum (c^h_k; 1 \leq k \leq s), \quad h = \begin{cases} 1, r=\overline{1,t} \text{ et } s=t, \\ 2, r=\overline{1,n} \text{ et } s=n. \end{cases}$$

$$C_{h0} \leq C'_{hr} + \sum (C^h_k; 1 \leq k \leq s), \quad h = \begin{cases} 1, r=\overline{1,n} \text{ et } s=t, \\ 2, r=\overline{1,n} \text{ et } s=n. \end{cases}$$

Remarque 3. Généralement la condition

$$(3) \quad c'_{hr} \leq C'_{hr}, \quad h = \begin{cases} 1 \text{ et } r=i=\overline{t,1}, \\ 2 \text{ et } r=j=\overline{n,1}. \end{cases}$$

T h é o r è m e. Une condition nécessaire et suffisante que dans un graphe escalier G^f (Fig. 1) existe une circulation (f, F) , où f est une orientation $\bar{b}/c' \bar{c}'/b$ ou $\bar{b}/c \bar{c}/b'$ et F une circulation de la forme (1) en satisfaisant (2) est que la condition (3) soit satisfaite.

N é c e s s i t é. De (1) nous avons

$$\bar{c}_s \leq \bar{F}_s \leq \bar{C}_s, \quad s=t, n;$$

$$c^h_{ks} \leq F^h_{ks} \leq C^h_{ks}, \quad s = \begin{cases} t \text{ et } h=1, k=1,2 \\ n \text{ et } h=2, k=1,2 \end{cases} \Rightarrow c'_{hs} \leq F^h_{hs} \leq C'_{hs};$$

$$\bar{F}_s = F^h_{1s} = F^h_{2s}, \quad s = \begin{cases} t \text{ et } h=1, k=1,2 \\ n \text{ et } h=2, k=1,2 \end{cases} \Rightarrow c'_{hs} \leq F^h_{hs} \leq C'_{hs}.$$

Nous avons

$$c^h_r \leq F^h_r \leq C^h_r,$$

$$c^h_{kr-1} \leq F^h_{kr-1} \leq C^h_{kr-1}, \quad r = \begin{cases} i, h=1 \text{ et } k=1,2, \\ j, h=2 \text{ et } k=1,2; \end{cases}$$

$$F^h_{kr-1} = F^h_{kr} + F^h_r,$$

et supposant

$$c'_{hr} \leq F^h_{kr} \leq C'_{hr}$$

il résulte (3) pour $h = \begin{cases} 1, r=i=\overline{t,1}, \\ 2, r=j=\overline{n,1} \end{cases}$ et finalement

$$\left. \begin{aligned} F_{k0} &= F_{k1}^h + F_1^h, \quad k=1,2, \quad h=1,2; \\ c_{k1}' &\leq F_{k1}^h \leq C_{k1}', \quad k=1,2, \quad h=1,2; \\ c_{k0} &\leq F_{k0} \leq C_{k0}, \quad k=1,2, \\ c_1^h &\leq F_1^h \leq C_1^h, \quad h=1,2, \end{aligned} \right\} \Rightarrow c_{k0}' \leq C_{k0}'.$$

Suffisance. Du (3) on définit

$$F_{10} = F_{20} = s_0^1 = s_0^2 \quad \text{avec} \quad c_{k0}' \leq s_0^k \leq C_{k0}', \quad k=1,2;$$

$$F^h = s^h \quad \text{avec} \quad \max(c_{hr}', s_{r-1}^h - C_r^h) \leq s_r^h \leq \min(C_{hr}', s_{r-1}^h - c_r^h),$$

pour

$$k=1,2 \quad \text{et} \quad r = \begin{cases} i=\overline{1,t}, & h=1, \\ j=\overline{1,n}, & h=2; \end{cases}$$

$$\bar{F}_s = \bar{F}_{ks}^h = s_s^h, \quad k=1,2 \quad \text{et} \quad h = \begin{cases} 1, & s=t, \\ 2, & s=n. \end{cases}$$

On vérifie par induction que les conditions (1) et (2) sont satisfaites; généralement, si

$$F_{kr}^h = s_r^h, \quad k=1,2 \quad \text{et} \quad r = \begin{cases} i=\overline{1,t}, & h=1, \\ j=\overline{1,n}, & h=2, \end{cases}$$

où

$$(4) \quad \max(c_{hr}', s_{r-1}^h - C_r^h) \leq s_r^h \leq \min(C_{hr}', s_{r-1}^h - c_r^h), \quad r = \begin{cases} i=\overline{1,t}, & h=1, \quad k=1,2, \\ j=\overline{1,n}, & h=2, \quad k=1,2, \end{cases}$$

on définit

$$F_r^h = s_{r-1}^h - s_r^h, \quad k=1,2, \quad h = \begin{cases} 1, & s=t \\ 2, & s=n \end{cases} \quad \text{et} \quad r = \begin{cases} i=\overline{1,t}, & h=1, \quad k=1,2, \\ j=\overline{1,n}, & h=2, \quad k=1,2; \end{cases}$$

$$\bar{F}_s = \bar{F}_{ks}^h = s_s^h, \quad k=1,2, \quad h = \begin{cases} 1, & s=t \\ 2, & s=n \end{cases} \quad \text{et} \quad r = \begin{cases} i=\overline{1,t}, & h=1, \quad k=1,2, \\ j=\overline{1,n}, & h=2, \quad k=1,2. \end{cases}$$

Les équations de conservation (1) sont évidemment satisfaites.

On vérifie par induction qu'existe (4); soit:

$$\max(c_{1r-1}^h, c_{hr}' + c_r^h, c_{2r-1}^h) = c_{hr-1}' \leq s_{r-1}^h \leq C_{hr-1}' = \min(C_{1r-1}^h, C_{hr}' + C_r^h, C_{2r-1}^h).$$

Il résulte

$$c_{hr}' + c_r^h \leq s_{r-1}^h \leq C_{hr}' + C_r^h,$$

ensuit

$$c_{hr}' \leq s_{r-1}^h - c_r^h, \quad s_{r-1}^h - C_r^h \leq C_{hr}'.$$

De

$$(5) \quad c_{hr}' \leq s_{r-1}^h - c_r^h, \quad c_{hr}' \leq C_{hr}', \quad \Rightarrow c_{hr}' \leq \min(C_{hr}', s_{r-1}^h - c_r^h).$$

De manière similaire, de

$$(6) \quad s_{r-1}^h - C_r^h \leq C_{hr}^/, \quad s_{r-1}^h - C_r^h \leq s_{r-1}^h - c_r^h \Rightarrow s_{r-1}^h - C_r^h \leq \min(C_{hr}^/, s_{r-1}^h - c_r^h).$$

De (5) et (6) il résulte

$$\max(c_{hr}^/, s_{r-1}^h - C_r^h) \leq \min(C_{hr}^/, s_{r-1}^h - c_r^h),$$

donc il existe (4).

Les conditions admissibles (2) sont satisfaites parce que de (4) nous avons

$$s_{r-1}^h - C_r^h \leq s_r^h \leq s_{r-1}^h - c_r^h, \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2. \end{cases}$$

D'ici il résulte

$$c_r^h \leq s_{r-1}^h - s_r^h = F_r^h \leq C_r^h.$$

Nous avons

$$\max(c_{1r-1}^h, c_{2r-1}^h) \leq c_{hr-1}^/ \leq s_{r-1} \leq C_{hr-1}^/ \leq \min(C_{1r-1}^h, C_{2r-1}^h), \quad r = \begin{cases} i = \overline{1, t}, & h = 1, \\ j = \overline{1, n}, & h = 2. \end{cases}$$

De manière similaire il résulte

$$\bar{c}_s \leq \bar{F}_s \leq \bar{C}_s, \quad s = \bar{t}, n.$$

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TIPURI DE CIRCULAȚII ADMISIBILE ÎNTR-UN GRAF SCARĂ NEORIENTAT (I)

(Rezumat)

Se studiază problema circulației admisibile de forma $\bar{b}/c' \bar{c}'/b$ [2] dintr-un graf scară neorientat și se găsește o condiție necesară și suficientă de existență.

ON SOME NEW INEQUALITIES RELATED TO THE ZEROS OF SOLUTIONS OF CERTAIN SECOND ORDER DIFFERENTIAL EQUATIONS (I)

BY

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1. Introduction. In this paper we are interested in obtaining results on the zeros of solutions of the second order differential equations of the forms

$$(A) \quad (r(t) | y(t) |^p | y'(t) |^{p-2} y'(t))' + q(t) | y(t) |^{2p-2} y(t) = 0,$$

$$(B) \quad (r(t) | y(t) |^{p-2} | y'(t) |^{p-2} y'(t))' + q(t) | y(t) |^{2p-4} y(t) = 0,$$

where $t \in I = [0, \infty)$ and I contains the points a and b ($a < b$); $p \geq 2$ is a real constant; the function $r: I \rightarrow \mathbb{R} = (-\infty, \infty)$ is C^1 -smooth and $r > 0$; the function $q: I \rightarrow \mathbb{R}$ is continuous.

The equations of this type arise in a mathematical description for one-dimensional turbulent flow of a polytropic gas in a porous medium (see [2], [6]). The general versions of Eqs. (A) and (B) have been recently dealt with in [2] and results on existence and uniqueness of the solutions are established. Here it will be assumed that every solution $y(t)$ of (A) or (B) exists on an interval I and is nontrivial.

A large number of papers have appeared in the literature which deals with the zeros of solutions of second order differential equations with different view points [4], [8], [10]...[12], [14], [16] and the references cited therein.

The main purpose of this paper is to derive some new inequalities which not only relates points a and b in I at which the solutions of (A) and (B) have zeros but also any point $c \in (a, b)$ where the solutions of (A) and (B) are maximized. The inequalities that we propose here can be used as handy tools in the study of the qualitative nature of the solutions of (A) and (B) for which earlier results on such inequalities do not apply. Here we give some applications to convey the importance of our results to the literature. In the second part of this work, we give further extensions of our results to the equations recently studied by the authors in [1], [3], [5], [7], [15] with different view points.

2. Statement of Results. Our main results which deals with the inequalities related to the zeros of solutions of differential equations (A) and (B) are given in the following theorems.

Theorem 1. Let $y(t)$ be a solution of (A) with $y(a)=y(b)=0$ and $y(t) \neq 0$ for $t \in (a, b)$. Let $|y(t)|$ be maximized in a point $c \in (a, b)$. Then

$$(1) \quad 1 \leq \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^b |q(s)| ds \right],$$

$$(2) \quad 1 \leq 2^p \left[\int_a^c r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^c |q(s)| ds \right],$$

$$(3) \quad 1 \leq 2^p \left[\int_c^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_c^b |q(s)| ds \right].$$

Theorem 2. Let $y(t)$ be a solution of (B) with $y(a)=y(b)=0$ and $y(t) \neq 0$ for $t \in (a, b)$. Let $|y(t)|$ be maximized in a point $c \in (a, b)$. Then

$$(4) \quad 1 \leq \frac{1}{3} \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^b |q(s)| ds \right],$$

$$(5) \quad 1 \leq \frac{1}{3} 2^p \left[\int_a^c r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^c |q(s)| ds \right],$$

$$(6) \quad 1 \leq \frac{1}{3} 2^p \left[\int_c^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_c^b |q(s)| ds \right].$$

Remark 1. We note that in the special cases when $r(t)=1$, inequalities (1)–(3) and (4)–(6) reduces respectively to the following inequalities:

$$(1') \quad 1 \leq (b-a)^{p-1} \int_a^b |q(s)| ds,$$

$$(2') \quad 1 \leq 2^p (c-a)^{p-1} \int_a^c |q(s)| ds,$$

$$(3') \quad 1 \leq 2^p (b-c)^{p-1} \int_c^b |q(s)| ds,$$

and

$$(4') \quad 1 \leq \frac{1}{3}(b-a)^{p-1} \int_a^b |q(s)| ds,$$

$$(5') \quad 1 \leq \frac{1}{3}2^p(c-a)^{p-1} \int_a^c |q(s)| ds,$$

$$(6') \quad 1 \leq \frac{1}{3}2^p(b-c)^{p-1} \int_c^b |q(s)| ds.$$

The inequalities (1') and (4') obviously yield the lower bounds on the distance between the consecutive zeros of the nontrivial solutions $y(t)$ of Eqs. (A) and (B) respectively, when $r(t)=1$, while the inequalities (2')–(3') and (5')–(6') relates the points a and b at which the solutions has zeros and also any point c between them where the solutions are maximized.

We next establish the following theorems which deal with the applications of some of our inequalities given above.

Theorem 3. Assume that

$$(7) \quad \int_0^\infty r^{-\frac{1}{p-1}}(s) ds < \infty,$$

$$(8) \quad \int_0^\infty |q(s)| ds < \infty.$$

Then every oscillatory solutions of (A) (and respectively of (B)) converges to zero as $t \rightarrow \infty$.

Theorem 4. Suppose that $|q(t)| \in L^\alpha[0, \infty)$, $1 \leq \alpha < \infty$. If $y(t)$ is any oscillatory solution of (A) with $r(t)=1$ (and respectively of (B) with $r(t)=1$), then the distance between consecutive zeros of $y(t)$ tends to infinity as $t \rightarrow \infty$.

Remark 2. In the special case when $p=2$, the equations (A) and (B) reduces respectively to the equations

$$(A') \quad (r(t)|y(t)|^2 y'(t))' + q(t)|y(t)|^2 y(t) = 0,$$

$$(B') \quad (r(t)y'(t))' + q(t)y(t) = 0.$$

The equation (B') has been studied by many authors in the literature, however, it seems to us, that very little is known about the equation of the form (A'). We note that, the finding of estimates of the type given in Theorems 1 and 2 are of great significance, not only for theory, but also gives the possibility to formulate a number of assertions qualitatively characterising the behavior of the solutions of Eqs. (A) and

(B) and their important special cases (A') and (B').

3. Proofs of Theorems 1 and 2. Let $M = \max |y(t)| = |y(c)|$, $c \in (a, b)$. By assumption in Theorem 1, M is a positive constant. Since $y(a) = y(b) = 0$, we have

$$(7) \quad M^2 = |y(c)|^2 = 2 \left| \int_a^c y(s) y'(s) ds \right| \leq 2 \int_a^c |y(s)| |y'(s)| ds,$$

$$(8) \quad M^2 = |y(c)|^2 = 2 \left| - \int_c^b y(s) y'(s) ds \right| \leq 2 \int_c^b |y(s)| |y'(s)| ds,$$

implying

$$(9) \quad M^2 \leq \int_a^b |y(s)| |y'(s)| ds = \int_a^b r^{-\frac{1}{p}}(s) r^{\frac{1}{p}}(s) |y(s)| |y'(s)| ds.$$

By taking p -th power on both sides of (9), applying Hölder's inequality with indices $p, p/(p-1)$, integrating by parts and using the fact that $y(t)$ is a solution of (A) such that $y(a) = y(b) = 0$, we have

$$(10) \quad \begin{aligned} M^{2p} &\leq \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^b r(s) |y(s)|^p |y'(s)|^p ds \right] = \\ &= \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^b (r(s) |y(s)|^p |y'(s)|^{p-2} y'(s)) y'(s) ds \right] = \\ &= \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[- \int_a^b (r(s) |y(s)|^p |y'(s)|^{p-2} y'(s)) y(s) ds \right] = \\ &= \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^b (q(s) |y(s)|^{2p-2} y(s)) y(s) ds \right] \leq \\ &\leq \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^b |q(s)| |y(s)|^{2p} ds \right] \leq \\ &\leq \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[M^{2p} \int_a^b |q(s)| ds \right]. \end{aligned}$$

Now dividing both sides of (10) by M^{2p} , we get (1).

Inequalities in (2) and (3) follows in a similar fashion, except that, now taking p -th power on both sides of (7) and (8) and applying Hölder's inequality with indices $p, p/(p-1)$, integrating by parts and using the fact that $y(t)$ is a solution of (A) such that $y(a)=y(b)=0$ and $y'(c)=0$. This completes the proof of Theorem 1.

By assumption of Theorem 2, we have the inequalities (7), (8) implying (9). Now, by taking p -th power on both sides of (9), applying Hölder's inequality with indices $p, p/(p-1)$, integrating by parts and using the fact that $y(t)$ is a solution of (B) such that $y(a)=y(b)=0$, we have

$$\begin{aligned}
 M^{2p} &\leq \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^b r(s) |y(s)|^p |y'(s)|^p ds \right] = \\
 &= \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^b (r(s) |y(s)|^{p-2} |y'(s)|^{p-2} y^2(s) y'(s) ds \right] = \\
 &= \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[- \int_a^b (r(s) |y(s)|^{p-2} |y'(s)|^{p-2} y'(s)) \frac{y^3(s)}{3} ds \right] = \\
 (11) \quad &= \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_a^b (q(s) |y(s)|^{2p-4} y(s)) \frac{y^3(s)}{3} ds \right] \leq \\
 &\leq \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\frac{1}{3} \int_a^b |q(s)| |y(s)|^{2p} ds \right] \leq \\
 &\leq \left[\int_a^b r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\frac{1}{3} M^{2p} \int_a^b |q(s)| ds \right].
 \end{aligned}$$

Now dividing both sides of (11) by M^{2p} , we get (4).

Inequalities (5) and (6) follow in a similar fashion, as indicated in the later part of the proof of Theorem 1. This completes the proof of Theorem 2.

4. Proofs of Theorems 3 and 4. Let $y(t)$ be an oscillatory solution of Eq. (A) on an interval I . Because of (7) and (8), we can choose $T > 0$ large enough so that for every $t \geq T$

$$(12) \quad \int_T^\infty r^{-\frac{1}{p-1}}(s) ds < \left(\frac{1}{2}\right)^{p-1}, \quad \int_T^\infty |q(s)| ds < 1.$$

First we shall show that every oscillatory solution $y(t)$ of (A) is bounded on I . Suppose to the contrary that $\limsup_{t \rightarrow \infty} |y(t)| = \infty$. Indeed, since $y(t)$ is oscillatory, there exists an interval (t_1, t_2) , and $M = \max\{|y(t)| : t_1 \leq t \leq t_2\}$. Choose c in (t_1, t_2) such that $|y(c)| = M$. Clearly, the inequality (1) in Theorem 1 is true on the interval (t_1, t_2) and taking into account (12), we have

$$\begin{aligned} 1 &\leq \left[\int_{t_1}^{t_2} r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_{t_1}^{t_2} |q(s)| ds \right] \leq \\ &\leq \left[\int_{t_1}^\infty r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_{t_1}^\infty |q(s)| ds \right] < \frac{1}{2}. \end{aligned}$$

This contradiction shows that every oscillatory solution $y(t)$ of (A) is bounded on I .

Next, we shall prove that $y(t) \rightarrow \infty$ as $t \rightarrow \infty$. If $y(t) \rightarrow 0$ as $t \rightarrow \infty$, then there exists a constant $d > 0$ such that

$$(13) \quad 2d < \limsup_{t \rightarrow \infty} |y(t)|.$$

Choose $T_0 \geq T$ such that

$$(14) \quad \int_{T_0}^\infty r^{-\frac{1}{p-1}}(s) ds < d^{-(p-1)}, \quad \int_{T_0}^\infty |q(s)| ds < d.$$

Because of (13) there exists $t_2 > t_1 > T_0$ such that $y(t_1) = y(t_2) = 0$ and

$$(15) \quad M_0 = \max\{|y(t)| : t_1 \leq t \leq t_2\} > d.$$

Clearly, the inequality (1) in Theorem 1 is true in the interval (t_1, t_2) and we have

$$(16) \quad 1 \leq \left[\int_{t_1}^{t_2} r^{-\frac{1}{p-1}}(s) ds \right]^{p-1} \left[\int_{t_1}^{t_2} |q(s)| ds \right].$$

Taking (14) into account, we obtain from (16) a contradiction that $1 < 1$. Therefore $y(t) \rightarrow 0$ as $t \rightarrow \infty$.

The proof of the case when $y(t)$ is an oscillatory solution of (B) follows in a similar fashion as given above, except that, here we will have to make use of the inequality (4) in place of the inequality (1). The proof of Theorem 3 is complete.

In order to prove Theorem 4, we first assume that $y(t)$ is any oscillatory solution of (A) with $r(t) = 1$ and the conclusion is not true. Then there exists a solution $y(t)$ with its sequence of zeros $\{t_n\}$ having a subsequence $\{t_{n_k}\}$ such that $|t_{n_{k+1}} - t_{n_k}| \leq$

$\leq N < \infty$ for all k . Let s_{n_k} be a point in $(t_{n_k}, t_{n_{k+1}})$ at which $|y(t)|$ is maximized. Then $|s_{n_k} - t_{n_k}| < N$ for all k . Let β be the index conjugate with α , namely $1/\alpha + 1/\beta = 1$. Suppose $|q(t)| \in L^\alpha[0, \infty)$, $1 \leq \alpha < \infty$, for k large enough we can write

$$(17) \quad \left[\int_{t_{n_k}}^{\infty} |q(s)|^\alpha ds \right]^{\frac{1}{\alpha}} \leq 2^{-p} N^{1-p-\frac{1}{\beta}}.$$

By using the inequality in (2'), we have

$$(18) \quad 1 \leq 2^p (s_{n_k} - t_{n_k})^{p-1} \int_{t_{n_k}}^{s_{n_k}} |q(s)| ds.$$

Now, using the Hölder's inequality with indices α, β on the right side of inequality (18), and then by making use of the inequality (17), we get

$$\begin{aligned} 1 &\leq 2^p (s_{n_k} - t_{n_k})^{p-1} \left[\int_{t_{n_k}}^{s_{n_k}} |q(s)|^\alpha ds \right]^{\frac{1}{\alpha}} (s_{n_k} - t_{n_k})^{\frac{1}{\beta}} \leq \\ &\leq 2^p (s_{n_k} - t_{n_k})^{p-1+\frac{1}{\beta}} \left[\int_{t_{n_k}}^{\infty} |q(s)|^\alpha ds \right]^{\frac{1}{\alpha}} < \\ &< 2^p N^{p-1+\frac{1}{\beta}} 2^{-p} N^{1-p-\frac{1}{\beta}} = 1, \end{aligned}$$

being a contradiction and the conclusion is true.

The proof of the case when $y(t)$ is any oscillatory solution of (B) with $r(t)=1$ follows in a similar fashion as given above by using the inequality (5'). This completes the proof of Theorem 4.

5. In the second part of the present work we shall consider the oscillatory behavior of solutions of other two types of second order differential equations which occur in a number of applications.

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ASUPRA UNOR NOI INEGALITĂȚI PRIVIND ZEROURILE SOLUȚIILOR UNOR ECUAȚII DIFERENȚIALE DE ORDINUL DOI (I)

(Rezumat)

Se stabilesc noi inegalități ce corelează punctele consecutive în care soluțiile ecuațiilor diferențiale de ordinul doi de tipul (A), (B) prezintă zerouri. Rezultatele sunt formulate în cadrul a patru teoreme.

SOME REMARKS ON THE LAPLACE EQUATION

$$xy'' + (ax+b)y' + (\lambda x + \mu)y = 0$$

BY

ADRIAN CORDUNEANU

0. We consider this equation under the hypothesis that a, b, λ, μ are real constants, and we shall search for it solutions defined for $x > 0$, using the method of the Laplace transform. Let us first remark that, if in the equation

$$(1) \quad xy'' + (ax+b)y' + (\lambda x + \mu)y = 0, \quad x > 0$$

we make the substitution $y = x^\alpha z$, where $z = z(x)$ stands for the new unknown function, we get

$$(2) \quad xz'' + [ax + (2-b)]z' + [\lambda x + \mu + a(1-b)]z = 0, \quad x > 0$$

by taking $\alpha = 1 - b$ ($b \neq 1$). This substitution is very useful in the case $b=2, \mu=a$ ([1], p. 28], because (2) takes the form

$$(2') \quad z'' + az' + \lambda z = 0,$$

that is, it becomes an equation with constant coefficients. We can now state

Proposition 1. *The general solution of the equation*

$$(3) \quad xy'' + (ax+2)y' + (\lambda x + a)y = 0, \quad x > 0$$

is given by $y = (1/x)z$, where $z = z(x)$ is the general solution of (2').

If $f = f(x)$ is an original function and $F = F(s)$ is its Laplace transform, we shall write $\mathcal{L}\{f(x)\} = F(s)$ or equivalently $f(x) = \mathcal{L}^{-1}\{F(s)\}$. Assume that (1) has a solution $y = y(x)$ which admits a Laplace transform (i.e. $y = y(x)$ is an original function) and that $y(0)$ is finite. Then, for its transform $Y = Y(s)$ we obtain the linear equation

$$(4) \quad Y'(s)(s^2 + as + \lambda) + Y(s)[(2-b)s + a - \mu] = y(0)(1-b).$$

Several situations may occur in the integration of (4), depending on the condition imposed to the coefficients a, b, λ, μ and the initial data $y(0)$.

1. Case i)

$$(5) \quad (b-2)s + \mu - a \equiv -n(s^2 + as + \lambda)', \quad n = \text{integer} \geq 1.$$

For the Eq. (1), under the condition $y(0)=0$, we obtain from (4) that

$$(6) \quad Y(s) = \frac{c_1}{(s^2 + as + \lambda)^n}, \quad c_1 = \text{arbitrary constant.}$$

If we put

$$(7) \quad \varphi_1(x) = \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + as + \lambda)^n} \right\}$$

we have a particular solutions for (1). Remarking that $b \neq 1$, we deduce that the general solution of the linear differential equation (4) is given by

$$(8) \quad Y(s) = \frac{c_1 + y(0)(1-b)P_{2n-1}(s)}{(s^2 + as + \lambda)^n},$$

where c_1 is an arbitrary constant and $P_{2n-1}(s)$ is the polynomial of degree $2n-1$, whose free term is zero, given by

$$(9) \quad P_{2n-1}(s) = \int (s^2 + as + \lambda)^{n-1} ds.$$

Putting

$$(10) \quad \varphi_2(x) = \mathcal{L}^{-1} \left\{ \frac{P_{2n-1}(s)}{(s^2 + as + \lambda)^n} \right\}$$

we obtain the following simple result

Proposition 2. Under condition (5), or the equivalent one

$$(11) \quad b = 2(1-n), \quad \mu = a(1-n), \quad n = \text{integer} \geq 1,$$

the general solutions of (1) is given by

$$(12) \quad y = c_1 \varphi_1(x) + c_2 \varphi_2(x),$$

where c_1, c_2 are arbitrary constants and φ_1, φ_2 are defined by (7) and (10), respectively.

Applying this result to Eq. (2), we conclude that under the condition

$$(13) \quad b = 2n, \quad \mu = an, \quad n = \text{integer} \geq 1$$

its general solution is $z = c_1 \varphi_1 + c_2 \varphi_2$, where φ_1 and φ_2 have the same meaning as above. We can now state

Proposition 3. Under condition (13), the general solution of (1) is given by

$$(14) \quad y = x^{1-2n} [c_1 \varphi_1(x) + c_2 \varphi_2(x)],$$

where c_1, c_2 are arbitrary constants and φ_1, φ_2 are defined by (7) and (10), respectively.

Taking into account that, in (11), we have $b=2m, \mu=am$ with $m=\text{integer} \leq 0$ and the preceding two Propositions, we conclude that for the equation

$$(15) \quad xy'' + (ax+2n)y' + (\lambda x+an)y = 0, \quad n = \text{integer},$$

it is possible to write down its general solution.

2. Case ii)

$$(16) \quad (b-2)s + \mu - a \equiv -n(s^2 + as + \lambda)^l, \quad n \neq \text{integer}, \quad n > 0.$$

Of course, (6) still holds. If we denote by s_1, s_2 ($s_1 \neq s_2$) the (real or complex) roots of the polynomial $s^2 + as + \lambda$, we have for (1) the particular solution

$$(17) \quad y(x) = x^{n-1} e^{s_1 x} * x^{n-1} e^{s_2 x} = e^{s_1 x} \int_0^x (x-t)^{n-1} t^{n-1} e^{-(s_1-s_2)t} dt, \quad x > 0,$$

where the sign $*$ stands for the convolution product of the two functions. The preceding equation was obtained from (6) with $c_1 = \Gamma(n)\Gamma(n) = \Gamma^2(n)$, Γ being the well-known Euler's function. Assuming now that $s_1 = s_2 = -a/2$, we obtain for (1) the particular solution

$$(18) \quad y = x^{2n-1} e^{-ax/2}, \quad x > 0,$$

which is valid without the restriction $n > 0$. We don't insist more since the case $s_1 = s_2$, or equivalently $a^2 - 4\lambda = 0$, will be considered later on.

3. Case iii)

$$(19) \quad a^2 - 4\lambda > 0, \quad \frac{(b-2)s + \mu - a}{s^2 + as + \lambda} = \frac{-m}{s-s_1} + \frac{-n}{s-s_2},$$

where $m, n = \text{integers} \geq 1$ and s_1, s_2 are the roots of the denominator. We remark that $m \neq n$ because for $m = n$ we would be in Case ii). We obtain from (4), with the initial condition $y(0) = 0$,

$$(20) \quad Y(s) = \frac{k}{(s-s_1)^m (s-s_2)^n}, \quad k = \text{an arbitrary constant},$$

which implies that φ_1 , defined by

$$(21) \quad \varphi_1(x) = \mathcal{L}^{-1} \left\{ \frac{1}{(s-s_1)^m (s-s_2)^n} \right\},$$

is a particular solution of (1). Remarking that, from the second condition (19) it follows that $b \neq 1$, we obtain the general solution of (4) under the form

$$(22) \quad Y(s) = \frac{c_1 + y(0)(1-b)P_{m,n}(s)}{(s-s_1)^m (s-s_2)^n},$$

where $P_{m,n}(s)$ is the polynomial of degree $m+n-1$, whose free term is zero, defined by

$$(23) \quad P_{m,n}(s) = \int (s-s_1)^{m-1} (s-s_2)^{n-1} ds.$$

Putting

$$(24) \quad \varphi_2(x) = \mathcal{L}^{-1} \left\{ \frac{P_{m,n}(s)}{(s-s_1)^m (s-s_2)^n} \right\},$$

we can state the following

Proposition 4. Under the hypotheses (19), the general solution of (1)

is given by $y=c_1\varphi_1(x)+c_2\varphi_2(x)$, where c_1, c_2 are arbitrary constants and φ_1, φ_2 are defined by (21) and (24), respectively.

Remark 1. If, in (19), $m>0$ and $n>0$ are not integers, we obtain from (20) only a particular solution $y=y(x)$ for (1), expressed by the convolution product

$$(25) \quad y(x) = x^{m-1} e^{sx} * x^{n-1} e^{sx}, \quad x > 0.$$

Remark 2. We can integrate (1) under a condition of type (19), i.e., if we have

$$(26) \quad a^2 - 4\lambda > 0, \quad \frac{(b-2)s + \mu - ab}{s^2 + as + \lambda} = \frac{-m}{s-s_1} + \frac{-n}{s-s_2},$$

where $m, n = \text{integers} \geq 1$.

4. Case iv)

$$(27) \quad a^2 - 4\lambda > 0, \quad \frac{(b-2)s + \mu - a}{s^2 + as + \lambda} = \frac{m}{s-s_1} - \frac{n}{s-s_2},$$

where m, n are integers, $n > m \geq 1$. It follows from (4), with $y(0)=0$, that

$$(28) \quad Y(s) = \frac{k(s-s_1)^m}{(s-s_2)^n}, \quad k = \text{an arbitrary constant},$$

what implies that

$$(29) \quad y(x) = \varphi_1(x) = e^{s_1 x} \frac{d^m}{dx^m} [x^{n-1} e^{-(s_1-s_2)x}]$$

is a particular solution of (1).

Imposing the stronger restriction $n \geq m+2$, which implies $b \neq 1$, we find that the general solution of (4) is given by

$$(30) \quad Y(s) = [c_1 + y(0)(1-b)Q_{m,n}] \frac{(s-s_1)^m}{(s-s_2)^n},$$

where

$$(31) \quad Q_{m,n}(s) = \int \frac{(s-s_2)^{n-1}}{(s-s_1)^{m+1}} ds.$$

Because we have

$$(32) \quad \frac{(s-s_2)^{n-1}}{(s-s_1)^{m+1}} = q(s) + \frac{A_1}{(s-s_1)^{m+1}} + \dots + \frac{A_{m+1}}{s-s_1},$$

where A_i are constants and $q(s)$ is a polynomial of degree $n-m-2$, we conclude that there exists a solution of (4) of the form

$$(33) \quad Y(s) = \left[q_1(s) + \frac{B_1}{(s-s_1)^m} + \dots + \frac{B_m}{s-s_1} + A_{m+1} \ln(s-s_1) \right] \frac{(s-s_1)^m}{(s-s_2)^n},$$

where $q_1'(s) = q(s)$ and B_i are constants, $i = \overline{1, m}$. We remark that $A_{m+1} \neq 0$.

From the equalities

$$(34) \quad \psi(x) = \int_x^\infty \frac{e^{-t}}{t} dt, \quad \mathcal{L}\{\psi(x)\} = \frac{1}{s} \ln(s+1),$$

we obtain, for every real s_1 , that

$$(35) \quad \mathcal{L}\{e^{(s_1+1)x} \psi(x)\} = \frac{1}{s-(s_1+1)} \ln(s-s_1).$$

Taking into account (33) and the condition $m \leq n-2$, we conclude that there exists an original function $y = \varphi_2(x)$, expressed by elementary functions, such that $\mathcal{L}\{\varphi_2(x)\}$ is the function $Y = Y(s)$, given by (33). To justify this assertion, it suffices to remark that the degree of $q_1(s)$ is $n-m-1$ and to write

$$(36) \quad \frac{(s-s_1)^m}{(s-s_2)^n} \ln(s-s_1) = \frac{(s-s_1)^m (s-(s_1+1))}{(s-s_2)^n} \frac{1}{s-(s_1+1)} \ln(s-s_1).$$

The counterimages of both functions in (36) are elementary functions, for every real s_1, s_2 and we can find a function having the Laplace transform (36), writing it as a convolution product. We can now state

Proposition 5. Under the hypothesis (27), where m and n are positive integers with $n \geq m+2$, the general solution of (1) is given by $y = c_1 \varphi_1(x) + c_2 \varphi_2(x)$, where c_1, c_2 are arbitrary constants, φ_1 is defined by (29) and φ_2 is the counterimage of (33).

Remark 3. The function φ_1 defined by (29) remains a solution of (1), even in the case $n \neq \text{integer}$, $m = \text{integer}$, $n > m \geq 1$.

Remark 4. We can integrate (1) if we have a condition of type (27) for the related equation (2).

5. Case v)

$$(37) \quad a^2 - 4\lambda = 0, \quad \mu = \frac{ab}{2}, \quad b \neq 1.$$

Under the supplementary and temporary condition $b \neq 2$, the general solution of (4) is given by

$$(38) \quad Y(s) = \frac{c_1}{\left[s + \frac{a}{2}\right]^{2-b}} + \frac{y(0)}{s + \frac{a}{2}}, \quad c_1 = \text{arbitrary constant},$$

which implies that (1) has the solutions

$$(39) \quad \varphi_1(x) = e^{-\frac{a}{2}x}, \quad \varphi_2(x) = x^{1-b} e^{-\frac{a}{2}x}.$$

For the moment we also assume that $b < 1$, what implies $\varphi_2(0) = 0$. But a simple proof shows that we may omit the restrictions $b \neq 2$ and $b < 1$, stating

Proposition 6. *Under the condition (37), the general solution of (1), i.e. of the equation*

$$(40) \quad xy'' + (ax+b)y' + \left[\frac{a^2}{4}x + \frac{ab}{2} \right] y = 0, \quad x > 0$$

is given by

$$(41) \quad y = e^{-\frac{a}{2}x} (c_1 + c_2 x^{1-b}),$$

where c_1, c_2 are arbitrary constants.

If we maintain the first two conditions in (37) and assume that $b=1$, we find for (1) the solution $\varphi_1(x)$ given by (39), only. Making the substitution $y = \varphi_1 z$, we can solve (1) as stated in

Proposition 7. *The general solution of the equation*

$$(42) \quad xy'' + (ax+1)y' + \left[\frac{a^2}{4}x + \frac{a}{2} \right] y = 0, \quad x > 0$$

is given by

$$(43) \quad y = e^{-\frac{a}{2}x} (c_1 + c_2 \ln x),$$

where c_1, c_2 are arbitrary constants.

Assuming $a^2 - 4\lambda = 0$, $b < 2$, $\mu - ab/2 = m > 0$ and $y(0) = 0$, we obtain from (4) the solution

$$(44) \quad Y(s) = \frac{k}{\left[s + \frac{a}{2} \right]^{2-b}} e^{-\frac{m}{s + \frac{a}{2}}}, \quad k = \text{arbitrary constant}.$$

Making use of the well-known formula

$$(45) \quad \mathcal{L} \left\{ x^{\frac{\nu}{2}} J_{\nu}(2\sqrt{x}) \right\} = \frac{1}{s^{(\nu+1)}} e^{-\frac{1}{s}} \quad (\nu \geq 0),$$

we easily deduce from (44) that the function

$$(46) \quad y = e^{-\frac{a}{2}x} x^{\frac{\nu}{2}} J_{\nu}(2\sqrt{mx}), \quad x > 0,$$

where $\nu = 1 - b \geq 0$ is a solution of the corresponding equation (1). Here, J_{ν} stands for the Bessel function of the first kind and index ν . We can now state the following

Proposition 8. Under the conditions $b \leq 1$ and $m > 0$, the function defined by (46), with $\nu = 1 - b$, is a solution of the equation

$$(47) \quad xy'' + (ax+b)y' + \left[\frac{a^2}{4}x + \frac{ab}{2} + m \right] y = 0, \quad x > 0.$$

Example. The function

$$(48) \quad y = e^{-\frac{a}{2}x} J_0(2\sqrt{x}) = e^{-\frac{a}{2}x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n$$

is a solution of

$$(49) \quad xy'' + (ax+1)y' + \left[\frac{a^2}{4}x + \frac{a}{2} + 1 \right] y = 0.$$

Remark 5. In the case $b > 0$ and $m > 0$, making the substitution $y = x^{1-b}z$ and using Proposition 8 for the related Eq. (2), we obtain that the function

$$(50) \quad y = e^{-\frac{a}{2}x} x^{-\frac{\nu}{2}} J_{\nu}(2\sqrt{mx}), \quad x > 0,$$

with $\nu = b - 1 > 0$, satisfies the Eq. (47).

Remark 6. In a similar way, we can find particular solutions for the equation

$$(51) \quad xy'' + (ax+b)y' + \left[\frac{a^2}{4}x + \frac{ab}{2} - m \right] y = 0, \quad x > 0,$$

in which $m > 0$, making use the formula

$$(52) \quad \mathcal{L} \left\{ x^{\frac{\nu}{2}} I_{\nu}(2\sqrt{x}) \right\} = \frac{1}{s^{\nu+1}} e^{\frac{1}{s}} \quad (\nu \geq 0),$$

where I_{ν} stands for the modified Bessel function of the first kind and index ν .

Example. The function

$$(53) \quad y = I_0(2\sqrt{x}) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n$$

satisfies the equation

$$(54) \quad xy'' + y' - y = 0 \quad (a=0, b=1, m=1).$$

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CÂTEVA OBSERVAȚII ASUPRA ECUAȚIEI LAPLACE

$$xy'' + (ax+b)y' + (\lambda x + \mu)y = 0$$

(Rezumat)

Se utilizează metoda transformatei Laplace și se studiază cazurile de integrabilitate, în funcție de felul rădăcinilor ecuației $s^2 + as + \lambda = 0$. În anumite situații, este avantajoasă substituția $y = x^\alpha z$ cu α convenabil ales și $z =$ funcția necunoscută.

GENERALIZED SOLUTIONS OF PARTIAL FUNCTIONAL-DIFFERENTIAL INCLUSIONS DEPENDING ON A PARAMETER

BY

GEORGETA TEODORU

1. Introduction. In this paper we consider a Darboux problem of the form

$$\frac{\partial^2 z}{\partial x \partial y} \in F \left[x, y, z(\cdot, \cdot), \frac{\partial z}{\partial x}(\cdot, \cdot), \frac{\partial z}{\partial y}(\cdot, \cdot), \frac{\partial^2 z}{\partial x \partial y}(\cdot, \cdot), \right. \\ \left. \frac{\partial^2 z}{\partial x \partial y}(\alpha_1(x, y), \beta_1(x, y)), \dots, \frac{\partial^2 z}{\partial x \partial y}(\alpha_\nu(x, y), \beta_\nu(x, y)), \lambda \right],$$

$z(x, 0) = f(x)$, $z(0, y) = g(y)$, $0 \leq x \leq a$, $0 \leq y \leq b$, where F is a multifunction taking its values in a separable Banach space X and λ is a parameter. We study the dependence of f , g , and λ on the set of generalized solutions (in the sense of J. Kisynski, [9]) of this problem.

This study was suggested by papers [11], [13]. The considered problem is more general than a similar problem of [13], since F also depends on ν additional arguments $z(\alpha_i(x, y), \beta_i(x, y))$, $i = \overline{1, \nu}$. Anyway, our paper is essentially based on [13] and therefore the reasonings and the results are similar to the ones thereof, with slight modifications. Moreover, we notice that we considered the same Darboux problem in the case when F does not depend on λ , in our paper [21], [22] using [11], [14]. The functions α_i and β_i , $i = \overline{1, \nu}$, occur in the evaluations imposed on F in a stating the sufficient conditions for the existence of solutions for every λ [11], [21], [22].

The consideration of this problem presents the advantage that it includes numerous particular cases, some of them being already studied [23], [24].

Let be the rectangle $Q = [0, a] \times [0, b]$, where a, b are two positive real numbers, with Lebesgue σ -algebra; $(X, \|\cdot\|)$ is a separable real Banach space, whose null element is denoted by Θ_X ; Λ is a non-empty set; F is a multifunction $F: Q \times X^{4+\nu} \times \Lambda \rightarrow 2^X$ with nonempty closed values; $p \in [1, +\infty[$.

If I is a compact real interval, we denote by $AC_p(I, X)$ the space of all strongly absolutely continuous functions $\varphi: I \rightarrow X$ such that $\varphi' \in L^p(I, X)$, where φ' is the strong derivative of φ . Given $f \in AC_p([0, a], X)$ and $g \in AC_p([0, b], X)$, with $f(0) = g(0)$, for fixed $\lambda \in \Lambda$, we consider the Darboux problem

$$(P_\lambda) \quad \begin{aligned} \frac{\partial^2 z}{\partial x \partial y} \in F & \left[x, y, z(\cdot, \cdot), \frac{\partial z}{\partial x}(\cdot, \cdot), \frac{\partial z}{\partial y}(\cdot, \cdot), \frac{\partial^2 z}{\partial x \partial y}(\cdot, \cdot), \frac{\partial^2 z}{\partial x \partial y}(\alpha_1(x, y), \beta_1(x, y)), \right. \\ & \left. \dots, \frac{\partial^2 z}{\partial x \partial y}(\alpha_\nu(x, y), \beta_\nu(x, y)), \lambda \right] \\ z(x, 0) &= f(x), \quad 0 \leq x \leq a, \\ z(0, y) &= g(y), \quad 0 \leq y \leq b. \end{aligned}$$

We define the notion of generalized solution in the sense of J. Kisynski, [9], [13].

A function $z: Q \rightarrow X$ is said to be a *generalized solution* of (P_λ) if there exists $\psi \in L^p(Q, X)$ such that

$$z(x, y) = f(x) + g(y) - f(0) + \int_0^x \int_0^y \psi(s, t) ds dt$$

for all $(x, y) \in Q$, and

$$\psi(x, y) \in F \left[x, y, z(x, y), f'(x) + \int_0^y \psi(x, t) dt, g'(y) + \int_0^x \psi(s, y) ds, \right. \\ \left. \psi(x, y), \psi(\alpha_1(x, y), \beta_1(x, y)), \dots, \psi(\alpha_\nu(x, y), \beta_\nu(x, y)), \lambda \right]$$

for almost every $(x, y) \in Q$, where the integrals are understood in the sense of Bochner and [13]

$$\int_0^x \int_0^y \psi(s, t) ds dt \quad \text{stands for} \quad \int_0^x \left[\int_0^y \psi(s, t) dt \right] ds.$$

In [22] we study the Darboux problem for the simpler case when $F: Q \times \mathbb{R}^n \times \Lambda \rightarrow 2^{\mathbb{R}^n}$, where we define the notion of solution imposing the condition that z is an absolutely continuous function in the Carathéodory sense, $z \in C^*(Q, \mathbb{R}^n)$, [1, § 565 — § 570], [3]—[7], which presents an analogy with the notion of generalized solution defined here.

Let $\Gamma_p(f, g, \lambda)$ be the set of all generalized solutions of (P_λ) . In this paper we study the multifunction $(f, g, \lambda) \rightarrow \Gamma_p(f, g, \lambda)$. Under suitable hypotheses, we establish its lower semicontinuity with respect to λ and its Lipschitzianity with respect to (f, g) (see Theorems 2.1 and 2.3).

1.1. Preliminaries [13]–[15]. Let Y, Z be two nonempty sets. A multifunction $\Phi: Y \rightarrow 2^Z$ is a function from Y in the family of all subsets of Z . A function $\varphi: Y \rightarrow Z$ is said to be a *selection* of Φ if $\varphi(y) \in \Phi(y)$ for all $y \in Y$. If $\Omega \subseteq Z$, we put $\Phi^-(\Omega) = \{y \in Y: \Phi(y) \cap \Omega \neq \emptyset\}$. If Y, Z are two topological spaces, we say that Φ is *lower semicontinuous* if, for every open set $\Omega \subseteq Z$, the set $\Phi^-(\Omega)$ is open in Y . When Y is first countable and Z is a metric space, Φ is lower semicontinuous if and only if, for every $\tilde{y} \in Y$, every sequence $\{y_n\}$ in Y converging to $\tilde{y}$ and every $\tilde{z} \in \Phi(\tilde{y})$, there exists a sequence $\{z_n\}$ in Z converging to $\tilde{z}$, such that $z_n \in \Phi(y_n)$ for all $n \in \mathbb{N}$ (see [8], Proposition 2.1). If $(Y, \mathcal{F})$ is a measurable space and Z is a topological space, we say that Φ is *measurable* if $\Phi^-(\Omega) \in \mathcal{F}$ for every open set $\Omega \subseteq Z$. When Φ is single-valued and Z is a separable metric space, this is equivalent to say that Φ is strongly measurable (see [2], p.61). If (Σ, δ) is a metric space, for every $x \in \Sigma$ and every pair of nonempty sets $A, B \subseteq \Sigma$, we put

$$\delta(x, A) = \inf_{z \in A} \delta(x, z), \quad \delta^*(A, B) = \sup_{z \in A} \delta(z, B)$$

and the Hausdorff-Pompeiu metric [20]

$$\delta_H(A, B) = \max\{\delta^*(A, B), \delta^*(B, A)\}.$$

Let $(Y, d), (Z, \rho)$ be two metric spaces. A multifunction $\Phi: Y \rightarrow 2^Z$, with nonempty values is said to be *Lipschitzian* if there exists a real number $L \geq 0$ (Lipschitz constant) such that

$$\rho_H(\Phi(y), \Phi(v)) \leq Ld(y, v)$$

for all $y, v \in Y$. If $L < 1$, we say that Φ is a *multivalued contraction*. Notice that any Lipschitzian multifunction is lower semicontinuous.

We recall the results of J. Kiszyński [9] related to generalized solutions [13].

1.2. Functions of class $C_p^*(Q, X)$ [13]. We denote by $C_p^*(Q, X)$ (see the space $L_p^*(\Delta)$ [9], p.88), the space of all continuous functions $z: Q \rightarrow X$ such that there exist $h \in L^p(Q, X)$, $k_1 \in L^p([0, a], X)$, $k_2 \in L^p([0, b], X)$ satisfying

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^{a-\varepsilon} \int_0^{b-\varepsilon} \|\varepsilon^{-2}[z(x+\varepsilon, y+\varepsilon) - z(x+\varepsilon, y) - z(x, y+\varepsilon) + z(x, y)] - h(x, y)\| dx dy = 0,$$

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^{b-\varepsilon} \|\varepsilon^{-1}[z(x+\varepsilon, b) - z(x, b)] - k_1(x)\| dx = 0,$$

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^{a-\varepsilon} \|\varepsilon^{-1}[z(a, y+\varepsilon) - z(a, y)] - k_2(y)\| dy = 0.$$

The following result is a particular case of Theorem 1 of [9, p.88] (see also [1, § 565 – § 570] for absolutely continuous functions, in the case $p = 1$).

Theorem 1.1. A function $z: Q \rightarrow X$ is an element of $C_p^*(Q, X)$ if and only if there exist $h \in L^p(Q, X)$, $h_1 \in L^p([0, a], X)$, $h_2 \in L^p([0, b], X)$ such that

$$z(x, y) = z(0, 0) + \int_0^x h_1(s) ds + \int_0^y h_2(t) dt + \int_0^x \int_0^y h(s, t) ds dt$$

for all $(x, y) \in Q$. Moreover, one has

$$\begin{aligned} \lim_{\varepsilon, \delta \rightarrow 0^+} \int_0^{a-\varepsilon} \int_0^{b-\delta} \|(\varepsilon\delta)^{-1} [z(x+\varepsilon, y+\delta) - z(x+\varepsilon, y) - z(x, y+\delta) + z(x, y)] - h(x, y)\|^p dx dy &= 0, \\ \lim_{\varepsilon \rightarrow 0^+} \int_0^{a-\varepsilon} \int_0^b \left\| \varepsilon^{-1} [z(x+\varepsilon, y) - z(x, y)] - h_1(x) - \int_0^y h(x, t) dt \right\|^p dx dy &= 0, \\ \lim_{\delta \rightarrow 0^+} \int_0^a \int_0^{b-\delta} \left\| \delta^{-1} [z(x, y+\delta) - z(x, y)] - h_2(y) - \int_0^x h(s, y) ds \right\|^p dx dy &= 0. \end{aligned}$$

1.3. The derivatives $\partial z/\partial x$, $\partial z/\partial y$, $\partial^2 z/\partial x \partial y$ of the function $z \in C_p^*(Q, X)$ [13], [9, p.91]. Let $z \in C_p^*(Q, X)$; we define the derivatives $\partial z/\partial x$, $\partial z/\partial y$, $\partial^2 z/\partial x \partial y$ of z as the functions of $L^p(Q, X)$ which satisfy the conditions

$$\begin{aligned} \lim_{\varepsilon, \delta \rightarrow 0^+} \int_0^{a-\varepsilon} \int_0^{b-\delta} \left\| (\varepsilon\delta)^{-1} [z(x+\varepsilon, y+\delta) - z(x+\varepsilon, y) - z(x, y+\delta) + z(x, y)] - \frac{\partial^2 z(x, y)}{\partial x \partial y} \right\|^p dx dy &= 0, \\ \lim_{\varepsilon \rightarrow 0^+} \int_0^{a-\varepsilon} \int_0^b \left\| \varepsilon^{-1} [z(x+\varepsilon, y) - z(x, y)] - \frac{\partial z(x, y)}{\partial x} \right\|^p dx dy &= 0, \\ \lim_{\delta \rightarrow 0^+} \int_0^a \int_0^{b-\delta} \left\| \delta^{-1} [z(x, y+\delta) - z(x, y)] - \frac{\partial z(x, y)}{\partial y} \right\|^p dx dy &= 0. \end{aligned}$$

From Theorem 1.1 it follows that every function $z \in C_p^*(Q, X)$ admits the derivatives $\partial z/\partial x$, $\partial z/\partial y$, $\partial^2 z/\partial x \partial y$ in the sense above specified, uniquely defined as elements of the space $L^p(Q, X)$ and a.e. in Q one has

$$\begin{aligned} \frac{\partial z(x, y)}{\partial x} &= h_1(x) + \int_0^y \frac{\partial^2 z(x, t)}{\partial x \partial t} dt, & \frac{\partial z(x, y)}{\partial y} &= h_2(y) + \int_0^x \frac{\partial^2 z(s, y)}{\partial s \partial y} ds, \\ (1) \quad \frac{\partial^2 z(x, y)}{\partial x \partial y} &= h(x, y). \end{aligned}$$

For every $z \in C_p^*(Q, X)$ we define as in [13]

$$\|z\|_{C_p^*(Q,X)} = \max_{(x,y) \in Q} \|z(x,y)\| + \left[\int_Q \left\| \frac{\partial z(x,y)}{\partial x} \right\|^p dx dy \right]^{1/p} + \left[\int_Q \left\| \frac{\partial z(x,y)}{\partial y} \right\|^p dx dy \right]^{1/p} + \left[\int_Q \left\| \frac{\partial^2 z(x,y)}{\partial x \partial y} \right\|^p dx dy \right]^{1/p}.$$

It is obvious that $\|\cdot\|_{C_p^*(Q,X)}$ is a norm on $C_p^*(Q,X)$. Moreover, the following result holds.

Proposition 1.1. *The space $C_p^*(Q,X)$ endowed with the norm $\|\cdot\|_{C_p^*(Q,X)}$ is complete (see the proof in [13]).*

For $\varphi \in AC_p(I, X)$, we define [13]

$$\|\varphi\|_{AC_p(I,X)} = \max_{t \in I} \|\varphi(t)\| + \left[\int_I \|\varphi'(t)\|^p dt \right]^{1/p}.$$

Obviously, $(AC_p(I,X), \|\cdot\|_{AC_p(I,X)})$ is complete.

Let us now put [13]

$$\Xi_p = \{(f,g) \in AC_p([0,a],X) \times AC_p([0,b],X) : f(0) = g(0)\}$$

and we consider on Ξ_p the norm of the graph; it is easy to verify that Ξ_p , endowed with this norm, is a closed linear subspace of $AC_p([0,a],X) \times AC_p([0,b],X)$.

We do not recall the other Propositions 1.2, 1.3, 1.4 of [13], which are also involved in the proof of our results.

2. Results. Since the theorems stated just below render to reformulations of the corresponding ones of [13] for the more general Darboux problem we have considered, we avoid giving their detailed proofs and we only present the necessary slight modifications.

Let us denote by d the metric on X induced by $\|\cdot\|$.

Theorem 2.1. *Let Λ be a first countable topological space. Assume that:*

i) *the multifunction $(x,y) \rightarrow F(x,y,z_1,z_2,\dots,z_{4+\nu},\lambda)$ is measurable for every $z_1,z_2,\dots,z_{4+\nu} \in X$, $\lambda \in \Lambda$;*

ii) *there exist $L \geq 0$, $M \in [0,1[$, $\bar{M} \in [0,1[$ such that, for every $z_i', z_i'' \in X$, $i = \overline{1,4+\nu}$, $\lambda \in \Lambda$ and for almost every $(x,y) \in Q$, one has*

$$d_H(F(x, y, z_1', z_2', \dots, z_{4+\nu}', \lambda), F(x, y, z_1'', z_2'', \dots, z_{4+\nu}'', \lambda)) \leq \\ \leq L \sum_{i=1}^3 \|z_i' - z_i''\| + M \|z_4' - z_4''\| + \bar{M} \sum_{i=1}^{\nu} \|z_{4+i}' - z_{4+i}''\|;$$

iii) the multifunction $\lambda \rightarrow F(x, y, z_1, z_2, \dots, z_{4+\nu}, \lambda)$ is lower semicontinuous for every $z_1, z_2, \dots, z_{4+\nu} \in X$ and for almost every $(x, y) \in Q$.

Then, the following assertions are equivalent:

i₁) for each convergent sequence $\{\lambda_n\}$ in Λ the set functions

$$A \rightarrow \int_A \left[d(\theta_X, F(x, y, \theta_X, \theta_X, \dots, \theta_X, \lambda_n)) \right]^p dx dy$$

are equi-absolutely continuous;

i₂) for every $(f, g) \in \Xi_p$, the multifunction $\lambda \rightarrow \Gamma_p(f, g, \lambda)$ is nonempty closed-valued and lower semicontinuous with respect to the norm topology on $C_p^*(Q, X)$.

For proof, fix $k > 0$ such that

$$\gamma = M + \bar{M}\nu + L \left[(ab)^{1-\frac{1}{p}} (kp)^{-\frac{1}{p}} + a^{1-\frac{1}{p}} + b^{1-\frac{1}{p}} \right] (kp)^{-\frac{1}{p}} < 1.$$

For every $(f, g) \in \Xi_p$, $\psi \in L^p(Q, X)$, $\lambda \in \Lambda$, put

$$\Phi(f, g, \psi, \lambda) = \{ \varphi \in L^p(Q, X) : \varphi(x, y) \in F(x, y, \eta(f, g, \psi)(x, y), \lambda) \text{ a.e. in } Q \},$$

where

$$\eta(f, g, \psi)(x, y) = \left(f(x) + g(y) - f(0) + \int_0^x \int_0^y \psi(s, t) ds dt, \right. \\ \left. f'(x) + \int_0^y \psi(x, t) dt, g'(y) + \int_0^x \psi(s, y) ds, \right. \\ \left. \psi(x, y), \psi(\alpha_1(x, y), \beta_1(x, y)), \dots, \psi(\alpha_\nu(x, y), \beta_\nu(x, y)) \right).$$

It is proved, as in [13], that $\Phi(f, g, \psi, \lambda) \neq \emptyset$ and $\Phi(f, g, \psi, \lambda)$ is closed, and for every $\lambda \in \Lambda$ the multifunction $\Phi(f, g, \cdot, \lambda)$ is a multivalued contraction on $L^p(Q, X)$ with Lipschitz constant γ and for each $\psi \in L^p(Q, X)$, the multifunction $\lambda \rightarrow \Phi(f, g, \psi, \lambda)$ is lower semicontinuous. It is also proved that the set of solutions $\Gamma_p(f, g, \lambda)$ is nonempty closed-valued and that the multifunction $\lambda \rightarrow \Gamma_p(f, g, \lambda)$ is lower semicontinuous using the same operator $T: L^p(Q, X) \rightarrow C_p^*(Q, X)$,

$$T(\varphi)(x, y) = \int_0^x \int_0^y \varphi(s, t) ds dt, \quad \varphi \in L^p(Q, X), \quad (x, y) \in Q,$$

defined in [13].

When the multifunction F does not depend on (x, y) , Theorem 2.1 gets the

following form.

Theorem 2.2. Suppose that F does not depend on (x, y) and assume that:
j) there exist $L \geq 0$, $M \in [0, 1]$, $\bar{M} \in [0, 1]$ such that, for every $z'_i, z''_i \in X$,
 $i = \overline{1, 4+\nu}$, $\lambda \in \Lambda$ one has

$$d_H(F(z'_1, z'_2, \dots, z'_{4+\nu}, \lambda), F(z''_1, z''_2, \dots, z''_{4+\nu}, \lambda)) \leq \\ \leq L \sum_{i=1}^3 \|z'_i - z''_i\| + M \|z'_4 - z''_4\| + \bar{M} \sum_{i=1}^{\nu} \|z'_{4+i} - z''_{4+i}\|;$$

jj) the multifunction $\lambda \rightarrow F(z_1, z_2, \dots, z_{4+\nu}, \lambda)$ is lower semicontinuous for every $z_1, z_2, \dots, z_{4+\nu} \in X$.

Then, for every $(f, g) \in \Xi_p$ the multifunction $\lambda \rightarrow \Gamma_p(f, g, \lambda)$ is nonempty closed-valued and lower semicontinuous with respect to the norm topology on $C_p^*(Q, X)$.

Theorem 2.3. Let F satisfy assumptions i) and ii) of Theorem 2.1. Further, assume that

iv) the real function $(x, y) \rightarrow d(\theta_X, F(x, y, \theta_X, \theta_X, \dots, \theta_X, \lambda))$ belongs to $L^p(Q, X)$ for all $\lambda \in \Lambda$.

Then there exists a constant c , depending only $a, b, p, L, M, \bar{M}$ and ν , such that, if D is the metric induced by $\|\cdot\|_{C_p^*(Q, X)}$, one has

$$D_H(\Gamma_p(f_1, g_1, \lambda), \Gamma_p(f_2, g_2, \mu)) \leq c \left[\|(f_1, g_1) - (f_2, g_2)\|_{\Xi_p} + \|f_1(0) - f_2(0)\| + \right. \\ \left. + \operatorname{ess\,sup}_{(x,y) \in Q} \sup_{\xi \in X^{4+\nu}} d_H(F(x, y, \xi, \lambda), F(x, y, \xi, \mu)) \right]$$

for every $(f_1, g_1), (f_2, g_2) \in \Xi_p$, $\lambda, \mu \in \Lambda$. The constant c is given by

$$c = \max \left\{ a^{\frac{1}{p}}, b^{\frac{1}{p}}, 1 \right\} + \frac{[ab]^{1-p\frac{1}{p}} + a + b + 1}{1 - \gamma} e^{k(a+b)} \max \left\{ (ab)^{\frac{1}{p}}, a^{\frac{1}{p}}, b^{\frac{1}{p}} \right\} (L+1).$$

Similarly with [13], from Theorems 2.1 and 2.3, it follows

Theorem 2.4. If hypotheses i), ii), iii) and i₁) of Theorem 2.1 hold, then the multifunction $(f, g, \lambda) \rightarrow \Gamma_p(f, g, \lambda)$ is nonempty closed-valued and lower semicontinuous with respect to the product topology on $\Xi_p \times \Lambda$ and the norm topology on $C_p^*(Q, X)$.

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SOLUȚII GENERALIZATE PENTRU INCLUZIUNI DIFERENȚIALE CU DERIVATE PARȚIALE CARE DEPEND DE UN PARAMETRU

(Rezumat)

Se consideră o problemă a lui Darboux de forma (P_λ) , unde F este o multifuncție cu valori într-un spațiu Banach separabil X , $F: Q \times X^{4+\nu} \times \Lambda \rightarrow 2^X$, $f: [0, a] \rightarrow X$, $g: [0, b] \rightarrow X$, $Q = [0, a] \times [0, b]$ și λ este un parametru care ia valori într-un spațiu topologic Λ . Se studiază dependența de f, g, λ a mulțimii soluțiilor generalizate (în sensul lui J. K i s y ř i s k i) $\Gamma_p(f, g, \lambda)$ ale acestei probleme și anume se arată că multifuncția $(f, g, \lambda) \rightarrow \Gamma_p(f, g, \lambda)$ este semicontinuu inferior în raport cu λ și Lipschitziană în raport cu (f, g) .

FINITE ELEMENT METHODS FOR SOME NONLINEAR PROBLEMS

BY

ARIADNA-LUCIA PLETEA

In paper [3] we have presented an abstract finite-element approximation for nonlinear equations generated by nonlinear operators with linear Gâteaux differential, K -symmetric and K -positive definite. In this paper we shall give some examples of problems for which we can apply the results that we have obtained.

The first example is a Sturm-Liouville nonlinear problem of odd degree, the second is a Sturm-Liouville problem of even degree and the third is a nonlinear problem studied by C i a r l e t [1], but from another point of view.

Let be the problem

$$(1) \quad P_1 u = \sum_{k=0}^{m-1} (-1)^k \frac{d^k}{dx^k} \left[p_{k+1} \left[\frac{d^{k+1} u}{dx^{k+1}} \right] \frac{d^{k+1} u}{dx^{k+1}} \right] = f,$$

$$f \in L^2((a, b)),$$

with conditions

$$(2) \quad u(a) = u'(a) = \dots = u^{(m-1)}(a) = 0,$$

$$u'(b) = u''(b) = \dots = u^{(m-1)}(b) = 0;$$

p_k are from C^k class, $k = \overline{1, m}$ and p_k depends only of $d^k u / dx^k$.

We impose the following conditions to the functions p_k , $k = \overline{1, m}$:

a) $d[tp_k(t)]/dt \geq c_k \geq 0$, $k = \overline{1, m}$

b) there is at least one index k_0 so that $d[tp_{k_0}(t)]/dt \geq c_{k_0} > 0$;

c) p_k , $k = \overline{1, m}$ are Lipschitzian functions.

The problem (1)+(2) with conditions (a)+(b) has a unique K -generalized solution $u \in L^2((a, b))$ [2].

The operator P_1 has Gâteaux differential

$$(3) \quad (DP_1)(u)h = \sum_{k=0}^{m-1} (-1)^k \frac{d^k}{dx^k} \left[p_{k+1} \left[\frac{d^{k+1} u}{dx^{k+1}} \right] + p' \left[\frac{d^{k+1} u}{dx^{k+1}} \right] \frac{d^{k+1} u}{dx^{k+1}} \right] \frac{d^{k+1} h}{dx^{k+1}},$$

linear in h and K -positive definite with $Ku = du/dx$. From this obtain that P_1 is strongly

K -monoton with $p=2$, according to definition [3], with $\gamma_1^2 = c_{k_0+1} [2/(b-a)^2]^{k_0+1}$ and $\gamma_2^2 = c_{k_0+1} [2/(b-a)^2]^{k_0}$.

The finite elements approximation is the solution of equation [3]

$$(4) \quad \sum_{k=0}^{m-1} \int_a^b p_{k+1} \left[\frac{d^{k+1} u_h}{dx^{k+1}} \right] \frac{d^{k+1} u_h}{dx^{k+1}} \frac{d^{k+1} v_h}{dx^{k+1}} dx = \int_a^b f \frac{dv_h}{dx} dx,$$

for all $v_h \in H_h$, where H_h is the finite element space, $H_h \subset L^2((a, b))$. H_h contains polynomials of degree k , where $k = (\sigma + 1)N_i$, σ is the maximum order of derivatives occurring in the defined reference finite element $\hat{T}$ and N_i is the number of nodes for the element T_i . From Lemma 1 [3] we have

$$(5) \quad \|u_h\| \leq \frac{1}{\gamma_1 \gamma_2} \|f\|.$$

P_1 is Lipschitz-continuous operator for bounded arguments according to condition c). So, we can apply Theorem 3 from [3] and we can obtain that

$$(6) \quad \|u - u_h\| \leq C \inf_{v_h \in H_h} \|K(u - v_h)\| \frac{1}{p-1}$$

and if

$$P^k(\hat{T}) \subset \hat{P} \subset H^0(\hat{T}),$$

$$H^{k+1}(\hat{T}) \subset C^\sigma(\hat{T})$$

and the weak solution of the problem (1)+(2) is in $H^{k+1}(a, b)$, then

$$\|u - u_h\|_{L^2(a, b)} \leq Ch^k |u|_{k+1, 2, (a, b)}.$$

The second problem is

$$(7) \quad P_2 u = \sum_{k=0}^m (-1)^k \frac{d^k}{dx^k} \left[p_k \left[\frac{d^k u}{dx^k} \right] \frac{d^k u}{dx^k} \right] = f,$$

$$(8) \quad f \in L^2((a, b)),$$

with conditions

$$u(a) = u'(a) = \dots = u^{(m-1)}(a) = 0,$$

$$(9) \quad u(b) = u'(b) = \dots = u^{(m-1)}(b) = 0.$$

We impose the same conditions to the function p_k as before.

We take $K=I$ so P_2 has Gâteaux differential linear in h and positive definite.

P_2 is strongly monotone and we have the same results as at first example.

The third example is

$$(10) \quad P_3 u = - \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left[|\delta_u|^{p-2} \frac{\partial u}{\partial x_i} \right] = f,$$

$f \in L^2(\Omega)$, Ω is a polygonal bounded set, $\Omega \subset \mathbb{R}^2$, $p \geq 2$ and

$$u=0 \quad \text{on } \partial\Omega.$$

We noted

$$|\delta u| = \sqrt{\left[\frac{\partial u}{\partial x_1}\right]^2 + \left[\frac{\partial u}{\partial x_2}\right]^2}.$$

We give some properties of this operator. Let be

$$P_3: W_0^{1,p}(\Omega) \rightarrow L^2(\Omega),$$

then

- a) P_3 is a potential operator and $(DP_3)(u)h$ Gâteaux differential linear in h ;
- b) P_3 is strongly K -monoton with $K=I$;
- c) P_3 is a Lipschitz continuous operator for bounded-arguments.

The finite element approximation is the solution of discrete problem

$$\int_{(\Omega)} |\delta u_h|^{p-2} (\delta u_h, \delta v_h) dx dy = \int_{\Omega} f v_h dx dy, \quad \forall v_h \in H_h,$$

H_h is the finite element space defined as in [1]. These solutions are bounded independently of H_h .

So we can apply Theorem 2 from [3] and we obtain the

Theorem. *If the solution $u \in W_0^{2,p}(\Omega)$, there exists a constant $C=C(\|f\|, \|u\|_{2,p,\Omega})$ so that*

$$\|u - u_h\|_{1,p,\Omega} \leq Ch^{\frac{1}{p-1}}.$$

Remark. If $u \in W_0^{1,p}(\Omega)$, then u_h converges to u in $L^2(\Omega)$ norma.

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METODA ELEMENTULUI FINIT PENTRU CÂTEVA PROBLEME NELINIARE

(Rezumat)

Se prezintă trei exemple de probleme neliniare cărora li se poate aplica metoda elementului finit în forma în care a fost prezentată abstract în lucrarea [3].

THE FINITE ELEMENT METHODS FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

BY

MARIA BECEANU and ARIADNA-LUCIA PLETEA

0. Introduction. Tricomi's equation

$$(0.1) \quad y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

represents the simplest model of degenerate elliptic equations in the upper half plane $y \geq 0$. It was M. V. K e l d y s h [1] who laid the foundation for further work by investigating the Dirichlet problem of two different types of degenerate elliptic equations:

$$(0.3) \quad y^m \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = 0,$$

$$(0.2) \quad \frac{\partial^2 u}{\partial x^2} + y^m \frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = 0,$$

($m > 0$, a , b , c real analytic functions near $y=0$) and gave the correct boundary value conditions.

Various generalizations and other boundary value problems were considered by S. G. M i h l i n [3], O. A. O l e i n i k [2].

1. Variational Formulation. Let be $D \subset \mathbb{R}^n$ a bounded and open subset with a continuous Lipschitz boundary. The problem is

$$(1.1) \quad A u = f,$$

where

$$(1.2) \quad A u = - \sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i} \left[a_{ij}(x) \frac{\partial u}{\partial x_j} \right] - \frac{\partial}{\partial x_n} \left[a_{nn}(x) \frac{\partial u}{\partial x_n} \right] + a_0(x) u,$$

with the conditions:

$$c_1) \quad a_{ij}(x) \in C^1(\bar{D}), \quad a_0(x) \in C^0(\bar{D}), \quad \forall i, j = \overline{1, n};$$

$$c_2) \quad a_{ij}(x) = a_{ji}(x), \quad \forall i, j = \overline{1, n};$$

$$c_3) \quad a_0(x) \geq c_0 > 0, \quad \forall x \in \bar{D};$$

$$c_4) \quad \text{the form } \sum_{i,j=1}^{n-1} a_{ij}(x) t_i t_j \text{ is positive definite};$$

$c_5) \quad \text{the coefficient } a_{nn}(x) \text{ is a nonnegative function and vanishes on a hypersurface } l \text{ lying on the plane } x_n = 0;$

$$c_6) \quad \partial D = \Gamma \cup l, \text{ where } \Gamma \text{ is situated in } x_n > 0.$$

We note $H = H^1(D)$ and $H_0 = \{u \in H^1(D); u|_{\Gamma} = 0\}$.

We define the bilinear form $a: H \times H \rightarrow \mathbb{R}$

$$(1.3) \quad a(u, v) = \int_D \left[\sum_{i,j=1}^{n-1} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_{nn}(x) \frac{\partial u}{\partial x_n} \frac{\partial v}{\partial x_n} + a_0(x) uv \right] dx$$

and the linear form $F: H \rightarrow \mathbb{R}$

$$(1.4) \quad F(v) = \int_D f(x) v(x) dx.$$

We shall look for a generalized solution of problem (1.1) – (1.2) and $c_1) - c_6)$.

This is a solution to the variational problem:

$$(P) \quad \text{let us find a } u \in H_0 \text{ so that } a(u, v) = (f, v), \quad \forall v \in H_0.$$

Theorem 1. *Let $f \in L^2(D)$ such that $f(x) = 0$ for $x_n < \delta$, where $x = (x_1, x_2, \dots, x_n) \in D$ and δ is a positive constant. We assume that the conditions $c_1), c_2), c_3), c_4), c_5)$ hold. Then the problem (P) has a unique weak solution $u \in H^1(D)$.*

Proof. We enclose D in some parallelepiped Π and assume without loss of generality that $\Pi = \{x | 0 < x_i < l_i, i = \overline{1, n}\}$ and let $D_\delta = \{x \in D | x = (x_1, x_2, \dots, x_n), x_n > \delta\}$, $\Pi_\delta = \{x \in \Pi | x = (x_1, x_2, \dots, x_n), x_n > \delta\}$.

Let $u \in H_0$. We represent $u(x)$ in the form

$$(1.5) \quad u(x', x_n) = - \int_{x_n}^{l_n} \frac{\partial u}{\partial t}(x', t) dt,$$

where $x' = (x_1, x_2, \dots, x_{n-1}) \in \Pi_n = \{x' | 0 < x_i < l_i, i = \overline{1, n-1}\}$ and we assume that $u(x) \equiv 0$ for $x \notin D$. If we square both sides of (1.5) and we estimate by Cauchy's inequality we have

$$u^2(x) \leq l_n \int_0^{l_n} \left[\frac{\partial u}{\partial x_n} \right]^2 dx_n.$$

Integrating over Π_δ we obtain

$$(1.6) \quad \int_{\Pi_\delta} u^2(x) dx \leq l_n^2 \int_{\Pi_\delta} \left(\frac{\partial u}{\partial x_n} \right)^2 dx \leq l_n^2 \sum_{i=1}^n \int_{D_\delta} \left(\frac{\partial u}{\partial x_i} \right)^2 dx.$$

In the domain D_δ there exists a positive constant μ_0 such that

$$\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + a_{nn}(x) \left(\frac{\partial u}{\partial x_n} \right)^2 \geq \mu_0 \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2.$$

From this it follows that

$$(1.7) \quad \begin{aligned} \int_{\Pi_\delta} u^2(x) dx &\leq \frac{l_n^2}{\mu_0} \int_{D_\delta} \left[\sum_{i,j=1}^{n-1} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + a_{nn}(x) \left(\frac{\partial u}{\partial x_n} \right)^2 \right] dx \leq \\ &\leq \frac{l_n^2}{\mu_0} \int_D \left[\sum_{i,j=1}^{n-1} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + a_{nn}(x) \left(\frac{\partial u}{\partial x_n} \right)^2 \right] dx. \end{aligned}$$

Because $f(x)=0$ for $x_n < \delta$ we have

$$(1.8) \quad |(f, u)|_{L^2(D)}^2 = \left| \int_{D_\delta} f(x) u(x) dx \right|^2 \leq \int_{D_\delta} f^2(x) dx \int_{D_\delta} u^2(x) dx.$$

Using (1.7) and (1.8) it follows

$$(1.9) \quad c_1 \|u\|_{H^1(D)}^2 \leq a(u, u) \leq c_2 \|u\|_{H^1(D)},$$

where $c_1 = \min\{\mu_0 / l_n^2, c_0\} > 0$ and c_2 is a positive constant.

For the existence and uniqueness of the solution to the problem (P) we shall use Lax-Milgram's Lemma. The hypotheses of this Lemma are satisfied. Evidently H_0 is a Hilbert space. The bounding of the bilinear form (1.3) results by using Hölder and Friedrichs-Poincaré's inequalities and the H_0 -ellipticity results from c_4 condition,

$$(1.10) \quad a(u, u) \geq c_1 \|u\|_{H_0}^2.$$

Then

$$(1.11) \quad \|u\|_{H^2(D)} \leq \frac{1}{c_1} \|f\|_{L^2(D)},$$

so the problem (P) has a unique solution.

Remark. If $u \in H^2(D)$ we can obtain the equivalence of problems (1.1)–(1.2), under conditions of the Theorem 1 and the problem (P). In the proof we use Green's formula, the conditions which satisfy the functions.

2. The Approximation of the Problem. An Abstract Evaluation of Error.

We consider the problem (P).

Let be $H_h, h \in \mathbb{N}, H \subset \mathbb{R}$, a space dimensionally finite, which ensures an internal

approximation of H_0 (the conforming finite element methods).

We associate the discrete problem to problem (P):

(P_h) find $u_h \in H_h$ so that $a(u_h, v_h) = (f, v_h)$, $\forall v_h \in H_h$.

We shall say that the family of discrete problems is convergent if, for any variational problem set in the space H_0 , we have

$$\lim_{h \rightarrow 0} \|u - u_h\| = 0 \text{ when } \|\cdot\| \text{ means the norm in } H_0.$$

The following theorem evaluates the error $\|u - u_h\|$ between the solutions of the problem (P) and (P_h).

Theorem 2 (Cea's Lemma). *There is a constant C, independent of the space H_h , so that*

$$\|u - u_h\| \leq C \inf_{v_h \in H_h} \|u - v_h\|.$$

So, a sufficient condition of convergence is the existence of a family (H_h) of subspaces of H_0 so that, for any $u \in H_0$,

$$\lim_{h \rightarrow 0} \inf_{v_h \in H_h} \|u - v_h\| = 0.$$

The proof of the theorem for problem (P) with the discrete variational problem (P_h) is the same as in [4].

3. Error Evaluation for the Problem on Polyedric Domains. We deal with the convergence of the solution u of the problem (P_h) introduced in the previous paragraph with the solution u of the corresponding problem (P).

Let be D a polyedric domain, subset of $\mathbb{R}^n$ and T_h an admissible triangulation of D so that $\bar{D} = \bigcup_{T \in T_h} T$. We associate the following dimensionally finite space to a triangulation T_h :

$$H_h = \{u_h \mid u_h \in C^0(D), u_h|_{\Gamma} = 0 \text{ and } \forall T \in T_h, u_h|_T \in P_k(T)\}.$$

Using a wellknown error evaluation due to Ciarlet [4] we obtain the following

Theorem 3. *Let be (T, Σ, P) , $T \in T_h$, $h \in \mathcal{H}$ a regular family of finite element methods of C^0 with its reference element $(\hat{T}, \hat{\Sigma}, \hat{P})$. We suppose there exists integer $k \geq 1$ so that*

$$P^k(\hat{T}) \subset \hat{P} \subset H^1(\hat{T}), H^{k+1} \hookrightarrow C^\sigma(\hat{T}),$$

where σ is the maximal order of partial derivatives occurring in the definitions of the set Σ .

Then if the solution $u \in H$ of the variational problem is also in the space $H^{k+1}(D)$, there exists a constant C independent of h such that

$$\|u - u_h\|_{1,D} \leq Ch^k |u|_{k+1,D}.$$

In Theorem 3 we have given assumptions which insure that $\|u - u_h\|_{1,D} = o(h^k)$ so that the error in the norm $\|\cdot\|_{0,D}$, i.e., the quantity $\|u - u_h\|_{0,D}$ is at least of the same order.

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METODA ELEMENTULUI FINIT PENTRU O CLASĂ DE ECUAȚII ELIPTICE DEGENERATE

(Rezumat)

Se aplică metoda elementului finit unei clase de ecuații eliptice care degenerază pe o porțiune a frontierei domeniului.

ON THE GENERATION AND DECAY OF PARALLEL FLOW IN A DUSTY FLUID UNDER CONSTANT PRESSURE GRADIENT

BY

K. SARKAR and A. K. GHOSH

1. Introduction. Although the channel flow problems associated with solid-liquid or solid-fluid systems are of primary importance to civil and hydraulic engineers yet such problems have many useful applications in vastly different branches of science. For example the engineering unit operations, such as classification, jigging, tabling, sedimentation, separation, contacting, fluidization are all concerned with the application of solid-fluid flows. Moreover, many natural phenomena such as, blood flow, flows in rocket tubes, dust in gas-cooling systems to enhance heat transfer processes, movement of inert particles in atmosphere and sand or other suspended particles in sea beaches are mainly associated with the dynamics of multiphase or two-phase fluid flow systems. Naturally, studies on these problems are mathematically interesting and physically useful.

There are two types of problems concerning the motion of fluid within a channel open or closed. In one such problems the flow is generated due to movement of the boundary walls while in the other the flow is caused by the variation of the pressure gradient. The unsteady channel flow problems of both the types concerning the motion of dust-free viscous fluid are found abundantly in the literature. Similar problems for visco-elastic liquids have already been investigated by Waters and King [1] while those for two-phase solid-fluid systems were considered by a number of investigators including Saffman [2], Michael and Miller [3], Michael [4], Liu [5], Yang and Healy [6], [7], Ghosh and Deb Nath [8]...[10], and Mitra and Bhattacharyya [11], [12]. In most of the papers, cited above, which are related to two-phase fluid flow systems, it is noticed that the unsteady motion is generated from rest due to the movement of the boundary walls.

In this paper attention is given to the generation and decay of the unsteady motion in a dusty fluid confined between two parallel plates when the flow takes place under a constant pressure gradient.

The present problem is concerned with the motion of a dusty fluid (two-phase fluid particle system) in a channel bounded by two parallel plates caused by a pressure gradient which is taken as constant. The corresponding decay problem when the

pressure gradient ceases to act have also been considered. The analysis is carried out to obtain the exact expressions for the fluid and the particle velocities. The wall shear stresses are also found. The effect of particles on the flow as well as on the shear stresses at the walls are analyzed quantitatively and shown graphically.

2. Mathematical Formulation of the Problem. We consider the flow of an incompressible viscous fluid containing suspension of dust particles in a channel bounded by two fixed parallel plates separated by a distance h . The x -axis is taken along the lower plate and y -axis in a direction normal to it. We suppose that the flow takes place under a constant pressure gradient. Following Saffman [2] along with the assumptions made by him on the dust particles, we write the equations of motion corresponding to our problem as

$$(2.1) \quad \frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} + \frac{k}{\tau} (v - u),$$

$$(2.2) \quad \frac{\partial y}{\partial t} = \frac{1}{\tau} (u - v),$$

where $u(y, t)$ and $v(y, t)$ are respectively the component of the fluid and the particle velocities in the direction of flow; k is the mass concentration of the particles, that is, the ratio of the mass density of the particles and the fluid density; τ is the relaxation time of the particles, which is small to make the dust fine and is larger for coarse dust. The other quantities p , ρ and ν have their usual meaning.

Since the motion generates from rest due to constant pressure gradient we have

$$(2.3) \quad \begin{aligned} \frac{\partial p}{\partial x} &= A = \text{constant}, \\ u(y, 0) &= v(y, 0) = 0, \quad 0 < y < h, \\ u(0, t) &= u(h, t) = 0, \quad t \geq 0. \end{aligned}$$

The problem now reduces to solve the set of equations (2.1) – (2.2) subject to the conditions given in (2.3).

3. Solution of the Problem. 3.1. Generating Motion. Introducing the non-dimensional quantities given by

$$\bar{u} = \frac{u}{u_0}, \quad \bar{v} = \frac{v}{v_0}, \quad \bar{y} = \frac{y}{h}, \quad \bar{t} = \frac{t}{h^2/\nu}, \quad \bar{\tau} = \frac{\tau}{h^2/\nu}$$

and $u_0 = -Ah^2/8\mu$ in Eqs. (2.1) and (2.2) and on dropping the bars we get

$$(3.1) \quad \frac{\partial u}{\partial t} = 8 + \frac{\partial^2 u}{\partial y^2} + \frac{k}{\tau} (v - u),$$

$$(3.2) \quad \frac{\partial v}{\partial t} = \frac{1}{\tau} (u - v),$$

with the conditions

$$(3.3) \quad \begin{aligned} u(0, t) &= u(1, t) = 0, \quad t > 0, \\ u(y, 0) &= v(y, 0) = 0, \quad t \leq 0. \end{aligned}$$

Applying Laplace transforms in Eqs. (3.1) to (3.3a) and then solving, the transformed solution for the fluid velocity comes out as

$$(3.4) \quad \bar{u} = 8 \frac{(1+s\tau)}{s^2(1+k+s\tau)} \left[1 - \frac{\sinh \alpha y + \sinh \alpha(1-y)}{\sinh \alpha} \right],$$

where $\alpha^2 = [s(1+k+s\tau)]/(1+s\tau)$ and s is the Laplace transform variable.

The Eq. (3.4) on inversion gives the fluid velocity in the form

$$(3.5) \quad u(y, t) = -4y(y-1) + 32 \sum_{n=1}^{\infty} \frac{D \sin Ny}{N},$$

where $D = D_1 + D_2$,

$$D_j = \frac{e^{\lambda_j t} (1 + \lambda_j \tau)^2}{\lambda_j [k + (1 + \lambda_j \tau)^2]}, \quad j = 1, 2,$$

$$N = (2n-1)\pi,$$

and

$$\lambda_1, \lambda_2 = \frac{1}{2\tau} \left\{ -(1+k+\tau N^2) \pm [(1+k+\tau N^2)^2 - 4\tau N^2]^{\frac{1}{2}} \right\}.$$

Using (3.5) in (3.2), the particle velocity is obtained as

$$(3.6) \quad v(y, t) = -4y(y-1)(1 - e^{-t/\tau}) + 32 \sum_{n=1}^{\infty} \frac{B \sin Ny}{N},$$

where $B = B_1 + B_2$,

$$B_j = \frac{(1 + \lambda_j \tau)(e^{\lambda_j t} - e^{-t/\tau})}{\lambda_j [k + (1 + \lambda_j \tau)^2]}, \quad j = 1, 2.$$

In particular, when $k=0$, the Eq. (3.5) becomes

$$(3.7) \quad u(y, t) = -4y(y-1) - 32 \sum_{n=1}^{\infty} \frac{e^{-N^2 t} \sin Ny}{N^3},$$

which is the well known result for the velocity distribution in a clean fluid (cf. [1]).

Finally, the dimensionless form of the shear stresses at the plates are

$$(3.8) \quad \tau_0 = -\frac{\partial u}{\partial y} \Big|_{y=0} = -4 - 32 \sum_{n=1}^{\infty} D,$$

$$\tau_1 = -\frac{\partial u}{\partial y} \Big|_{y=1} = 4 + 32 \sum_{n=1}^{\infty} D,$$

where D is given in (3.5).

Clearly, the magnitudes of the shear stress on both the plates are the same but one acts in a direction opposite to that of the other, which is an expected result.

3.2. Decay of Steady Motion. We consider that the steady motion due to the action of a constant pressure gradient on the flow goes on decaying when the pressure gradient ceases to act.

In this situation we find that the transformed equation for the determination of the fluid velocity takes the form

$$(3.9) \quad \frac{d^2 \bar{u}}{dy^2} - \alpha^2 \bar{u} = 4y(y-1) \frac{\alpha^2}{s},$$

which is to be solved subject to the conditions

$$(3.10) \quad \bar{u}(0, s) = \bar{u}(1, s) = 0.$$

The explicit form of $\bar{u}$ is given by

$$(3.11) \quad \bar{u} = -\frac{4}{s}y(y-1) - \frac{8}{\alpha^2 s} \left[1 - \frac{\sinh \alpha y + \sinh \alpha(1-y)}{\sinh \alpha} \right],$$

which on inverting provides the result for the fluid velocity in decaying motion as

$$(3.12) \quad u(y, t) = -32 \sum_{n=1}^{\infty} \frac{D \sin Ny}{N},$$

where D is given in (3.5). In particular, when $k=0$, the clean fluid velocity in this situation becomes

$$(3.13) \quad u(y, t) = 32 \sum_{n=1}^{\infty} \frac{e^{-N^2 t} \sin Ny}{N^3}.$$

Further the particle velocity during decay of the fluid motion is found as

$$(3.14) \quad v(y, t) = -32 \sum_{n=1}^{\infty} \frac{B \sin Ny}{N},$$

where B is given in (3.6)

Finally, the shear stress at the walls corresponding to decay problem is given by

$$(3.15) \quad \tau_0 = -32 \sum_{n=1}^{\infty} D, \quad \tau_1 = 32 \sum_{n=1}^{\infty} D,$$

where D is given in (3.5).

4. Discussions. 1. To find the effect of the particles on the fluid motion in both the cases of generating and decay problem, the results (3.5) and (3.12) are evaluated for different values of k when $\tau=0.1$. These results are shown graphically in Figs. 1a and 1b respectively.

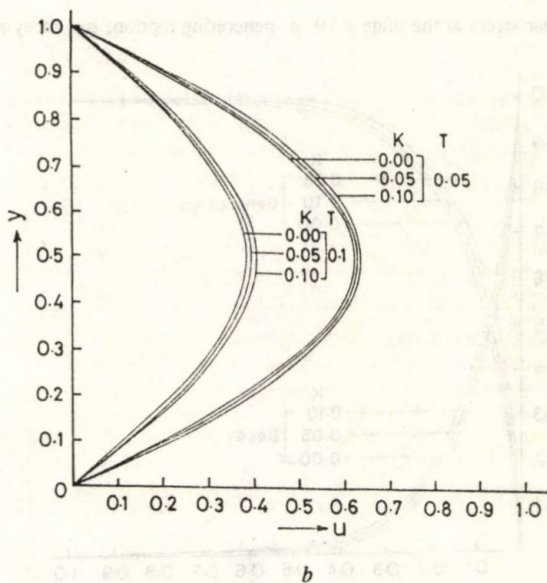
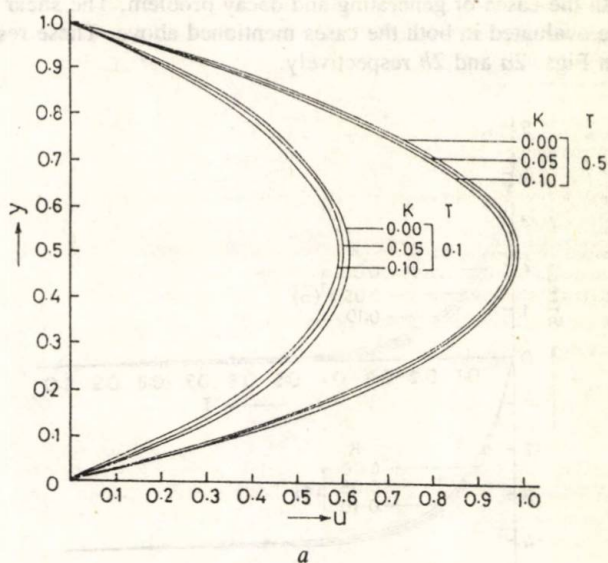


Fig. 1 — Dusty fluid velocity for different values of k , when $\tau=0.1$:
 a — in the generating motion; b — in the decaying motion.

2. It is observed that an increase in particle concentration k decelerates the

motion in both the cases of generating and decay problem. The shear stresses on the plate $y=0$ are evaluated in both the cases mentioned above. These results are shown graphically in Figs. 2a and 2b respectively.

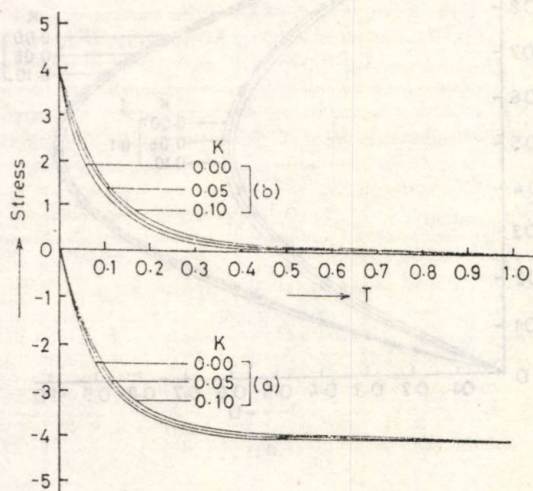


Fig. 2 — Shear stress at the plate $y=0$: a — generating motion; b — decaying motion.

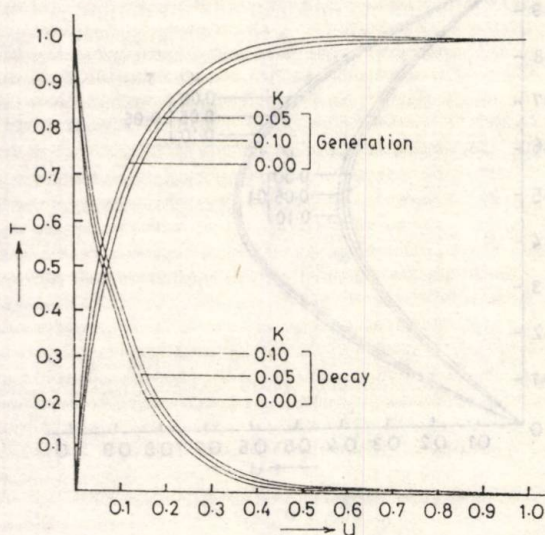


Fig. 3 — Central line flow ($y=0.5$) for different values of k .

3. The magnitude of the stress is found to decrease due to the presence of

particles in the generating motion. However, a reverse effect is observed in the decay problem in which the presence of particles increases the magnitude of the shear stress at the plate.

4. Finally, the variation of the central line flow ($y=0.5$) due to the presence of particles are shown in Fig. 3 for both the cases.

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ASUPRA GENERĂRII ȘI STINGERII CURGERII PARALELE A UNUI FLUID CU PARTICULE ÎN SUSPENSIE SUPUS UNUI GRADIENT DE PRESIUNE CONSTANT

(Rezumat)

Se studiază curgerea unui fluid "prăfos" printr-un canal, produsă de un gradient de presiune constant, și stingerea curgerii când gradientul de presiune dispăre. Se dau soluțiile exacte, în ambele cazuri, atât pentru fluid cât și pentru vitezele particulelor în suspensie. Se analizează cantitativ și se reprezintă grafic efectul particulelor asupra curgerii precum și asupra tensiunilor la nivelul pereților canalului.

**FINITE ELEMENT METHOD IN ANALYSIS OF TWO-DIMENSIONAL
MHD SYSTEMS OSCILLATIONS**

BY

GH. PROCOPIUC

1. Introduction. Computational fluid dynamics is becoming an important engineering tool. Simulations of fluid flows began in the early 60's with potential flows, first incompressible or compressible hypersonic, then compressible transonic flows. The calculations were done using finite difference or panel methods. The 70's saw the first implementation of the finite element method for the potential equation and the Navier-Stokes equations. Recent years have seen the development of faster algorithms for the treatment of problems which are more and more complex [1].

The goal of this paper is to analyse how the finite element method can be applied to obtain an approximation of solution of the system of differential equations of the plane magnetohydrodynamic oscillations.

In numerical solving of our problem using the finite element method, it appears the following difficulty: to each time step, the variable boundary passes over the nodes of the mesh, what complicates the discretization of equations in the neighbourhood of boundary. This difficulty exists a fortiori for the problem with free boundary, when one looks for determination of the free boundary position as a function of time. We use here a method of numerical solution described in [2], which avoids this disadvantage and which is based on the utilization of finite elements relating to space and time. These elements allow the variation of the nodes of the mesh to each step in time and then can be followed the free boundary.

2. Oscillations Equations of Two-dimensional Magnetohydrodynamics Systems. Let us suppose that in an ideal isothermic nonhomogeneous fluid in stationary equilibrium in an electromagnetic field it arises at a given moment an infinitesimal disturbance. Consequently, the density ρ , the pressure p and the magnetic field $\mathbf{B}$, corresponding to the stationary equilibrium, will be changed with the small values $\delta\rho$, δp , $\delta\mathbf{B}$. For these quantities, in nonhomogeneous case, from the magnetohydrodynamic equations in the linear approach relative to the disturbances, it

results the following equations [3]:

$$\begin{aligned}
 (2.1) \quad & \frac{\partial(\delta p)}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \\
 & \rho \frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad} p + \frac{1}{\mu_0} \mathbf{B} \times (\operatorname{curl}(\delta \mathbf{B})) + \frac{1}{\mu_0} (\delta \mathbf{B}) \times \operatorname{curl} \mathbf{B} = 0, \\
 & \frac{\partial(\delta \mathbf{B})}{\partial t} - \operatorname{curl}(\mathbf{v} \times \mathbf{B}) = 0, \\
 & \operatorname{div}(\delta \mathbf{B}) = 0, \quad \delta p = c_0^2 \delta \rho.
 \end{aligned}$$

In this survey we shall be interested in study of the plane oscillations of the fluid. Therefore, we admit that the undisturbed state — of equilibrium — is characterized by

$$\mathbf{B} = (B_x, B_y, 0), \quad B_x = B_x(x, y), \quad B_y = B_y(x, y),$$

$$p = p(x, y), \quad \rho = \rho(x, y),$$

which are constrained to satisfy the condition of equilibrium

$$\operatorname{grad} p = \frac{1}{\mu_0} \mathbf{B} \times \operatorname{curl} \mathbf{B} = 0,$$

with $\operatorname{div} \mathbf{B} = 0$. In this case, the Eqs. (2.1) become

$$\begin{aligned}
 & \frac{\partial}{\partial t}(\delta \rho) + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) = 0, \\
 & \frac{\partial}{\partial t}(\rho v_x) + \frac{\partial}{\partial x} \left[c_0^2 \delta \rho + \frac{B_y}{\mu_0} \delta B_y \right] - \frac{\partial}{\partial y} \left[\frac{B_y}{\mu_0} \delta B_x \right] + \frac{1}{\mu_0} \frac{\partial B_y}{\partial y} \delta B_x - \frac{1}{\mu_0} \frac{\partial B_x}{\partial y} \delta B_y = 0, \\
 & \frac{\partial}{\partial t}(\rho v_y) - \frac{\partial}{\partial x} \left[\frac{B_x}{\mu_0} \delta B_y \right] + \frac{\partial}{\partial y} \left[c_0^2 \delta \rho + \frac{B_x}{\mu_0} \delta B_x \right] - \frac{1}{\mu_0} \frac{\partial B_y}{\partial x} \delta B_x + \frac{1}{\mu_0} \frac{\partial B_x}{\partial x} \delta B_y = 0, \\
 & \frac{\partial}{\partial t}(\delta B_x) - \frac{\partial}{\partial y} \left[\frac{B_y}{\rho} \rho v_x - \frac{B_x}{\rho} \rho v_y \right] = 0, \\
 & \frac{\partial}{\partial t}(\delta B_y) + \frac{\partial}{\partial x} \left[\frac{B_y}{\rho} \rho v_x - \frac{B_x}{\rho} \rho v_y \right] = 0,
 \end{aligned}$$

and they can be written in the form

$$(2.2) \quad \frac{\partial}{\partial t} \mathbf{U} + \frac{\partial}{\partial x} (A \mathbf{U}) + \frac{\partial}{\partial y} (B \mathbf{U}) + C(\mathbf{U}) = 0,$$

where $\mathbf{U} = (\delta \rho, \rho v_x, \rho v_y, \delta B_x, \delta B_y)$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ c_0^2 & 0 & 0 & 0 & \frac{B_y}{\mu_0} \\ 0 & 0 & 0 & 0 & -\frac{B_x}{\mu_0} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{B_y}{\rho} & -\frac{B_x}{\rho} & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{B_y}{\mu_0} & 0 \\ c_0^2 & 0 & 0 & \frac{B_x}{\mu_0} & 0 \\ 0 & -\frac{B_y}{\rho} & \frac{B_x}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu_0} \frac{\partial B_y}{\partial y} & -\frac{1}{\mu_0} \frac{\partial B_x}{\partial y} \\ 0 & 0 & 0 & -\frac{1}{\mu_0} \frac{\partial B_y}{\partial x} & \frac{1}{\mu_0} \frac{\partial B_x}{\partial x} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathfrak{R} = \{(x, y, t), 0 < x < a(t), 0 < y < b(t), t > 0\}$ be a tridimensional domain, where $\mathbf{a}(t) = (a(t), b(t))$ is an unknown function.

Let us associate, to the evolution equation (2.2), the initial data

$$(2.3) \quad \mathbf{U}(x, y; 0) = \mathbf{U}_0(x, y), \quad \text{for } 0 \leq x \leq a_0, \quad 0 \leq y \leq b_0,$$

with a_0 and b_0 — two positive numbers and $\mathbf{U}_0: [0, a_0] \times [0, b_0] \rightarrow \mathbb{R}^5$ a given function, and the boundary conditions in the form

$$(2.4) \quad g_x(\mathbf{U}(x, 0; t), t) = 0, \quad \text{for } x = 0 \text{ or } x = a(t), t > 0,$$

$$(2.5) \quad g_y(\mathbf{U}(0, y; t), t) = 0, \quad \text{for } y = 0 \text{ or } y = b(t), t > 0,$$

where $g_x, g_y: \mathbb{R}^5 \times \mathbb{R} \rightarrow \mathbb{R}$ are two given functions.

Finally, the velocity of displacement of the free boundary is given by the differential equation

$$(2.6) \quad \frac{d\mathbf{a}}{dt} = \mathbf{F}(\mathbf{U}(\mathbf{a}(t), t), t),$$

with the initial condition

$$(2.7) \quad \mathbf{a}(0) = \mathbf{a}_0,$$

where $\mathbf{a}_0 = (a_0, b_0)$ and $\mathbf{F}: \mathbb{R}^5 \rightarrow \mathbb{R}^2$ is a given function.

If the coefficients A, B, C and the initial boundary conditions are sufficiently smooth functions, the problem (2.2)–(2.7) has a unique classical solution.

In the less restricted constraints for the coefficients and initial and boundary

conditions, we can introduce the notion of *weak solution* for the problem (2.2)–(2.7).

Let $\mathcal{L}$ be a linear differential operator defined by

$$(2.8) \quad \mathcal{L}(U) = \frac{\partial}{\partial t} U + \frac{\partial}{\partial x} (AU) + \frac{\partial}{\partial y} (BU) + CU$$

and $\mathcal{L}^*$ its formal adjoint

$$(2.8') \quad \mathcal{L}^*(U) = -\frac{\partial}{\partial t} U - {}^tA \frac{\partial}{\partial x} (U) - {}^tB \frac{\partial}{\partial y} (U) + {}^tCU.$$

A solution to the equation (2.2) satisfies the integral identity

$$(2.9) \quad A_G(U, \Phi) = \int \int \int_G \mathcal{L}(U) \Phi \, dx dy dt = 0,$$

for all sufficiently smooth functions $\Phi: \mathbb{R}^3 \rightarrow \mathbb{R}^5$, where

$$G = \{(x, y, t) \mid 0 < x < a(t), 0 < y < b(t), \tau_1 < t < \tau_2\}, \quad 0 < \tau_1 < \tau_2.$$

Integrating by parts and using the Gauss-Ostrogradski theorem, (2.9) can be written as

$$(2.9') \quad \begin{aligned} A_G(U, \Phi) = & \int \int \int_G U \mathcal{L}^*(\Phi) \Phi \, dx dy dt + \\ & + \int \int_{\partial G} \Phi (U dx dy + A U dy dt + B U dt dx) = 0. \end{aligned}$$

The integral identity (2.9') enables us to define the weak solutions to the system (2.2). On the other hand, taking for Φ arbitrary constant functions, one obtains the relation

$$(2.10) \quad \int \int \int_G C U dx dy dt + \int \int_{\partial G} U dx dy + A U dy dt + B U dt dx = 0.$$

3. Analysis by Finite Element Method of MHD Oscillations System with Free Boundary. In what follows, we describe a numerical method to solve the problem (2.2) – (2.7). We deal first with the system (2.2) in the initial and boundary conditions (2.3) – (2.5) under the hypothesis that $a(t)$ is a continuous given function. Then, we describe the computing of the free boundary, i.e. of the function $a(t)$.

Let us discretize the domain $\mathfrak{R}$ using parallelipipedic curvilinear elements. Let I, J denote two positive integers. We consider the tridimensional (plane-temporal) mesh of points $P_{ij}^n(x_i^n, y_j^n, t^n)$ belonging to $\mathfrak{R}$ with $i=\overline{0, I}, j=\overline{0, J}$ and

$$t^0 = 0, \quad t^n < t^{n+1};$$

$$x_i^n = 0, \quad x_i^n < x_{i+1}^n \quad \text{and} \quad x_I^n = a(t^n),$$

$$y_j^n = 0, \quad y_j^n < y_{j+1}^n \quad \text{and} \quad y_J^n = b(t^n).$$

We denote by K_{ij}^n the parallelepiped with vertices

$$P_{ij}^n, \quad P_{i+1,j}^n, \quad P_{i+1,j+1}^n, \quad P_{i,j+1}^n,$$

$$P_{ij}^{n+1}, \quad P_{i+1,j}^{n+1}, \quad P_{i+1,j+1}^{n+1}, \quad P_{i,j+1}^{n+1}.$$

Finally, let be

$$G_h^n = \bigcup_{i,j=0}^{I,J} K_{ij}^n \quad \text{and} \quad R_h = \bigcup_{n \geq 0} G_h^n,$$

h being a parameter characterizing the mesh and vanishing at the refinement of the mesh.

Let V_h be the space of continuous functions which are defined on R_h and having restrictions to K_{ij}^n linear on each edge and linear in x and y for each fixed t . All these functions are uniquely in terms of their values in the nodes of the mesh.

For each $i, k \in \{1, \dots, I\}$ and $j, l \in \{1, \dots, J\}$ we then construct the functions

$$\phi^{(kl)}(P_{ij}^n) = \delta_{ik} \delta_{jl}, \quad n \geq 0$$

and let us denote by E_h the subspace of V_h generated by the functions $\phi^{(kl)}$, $k = \overline{1, I}$, $l = \overline{1, J}$.

We shall approximate the solution U through the vector function U_h having the components in the space V_h , i.e. $U_h \in V_h^5$.

Let $A^n(U_h, \Phi_h)$ be an approximation of the bilinear form $A_G(U, \Phi)$ defined by (2.9'), obtained by substituting G through G_h^n and calculating the integrals by means of numerical integration. It then gives the following discrete problem, where Y_h is a subspace of E_h^5 depending on the type of boundary conditions:

Find $U_h \in V_h^5$, satisfying conditions:

$$(3.1) \quad A^n(U_h, \Phi_h) = 0, \quad \forall \Phi_h \in Y_h, \quad n \geq 0;$$

$$(3.2) \quad U_h(P_{ij}^0) = U_0(x_i, y_j), \quad i = \overline{0, I}, \quad j = \overline{0, J};$$

$$(3.3) \quad g_x(U_h(P_{ij}^n), t^n) = 0, \quad i = \overline{1, I-1}, \quad j \in \{0, J\}, \quad n > 0,$$

$$(3.4) \quad g_y(U_h(P_{ij}^n), t^n) = 0, \quad i \in \{0, I\}, \quad j = \overline{0, J-1}, \quad n > 0.$$

For each temporal step, the position of free boundary in the following step is calculated by discretization of the problem (2.6) – (2.7), e.g. using the formula

$$a^{n+1} = a^n + \frac{1}{2}(t^{n+1} - t^n)(F_{IJ}^n + F_{IJ}^{n+1}),$$

where $F_{IJ}^n = F(U_{IJ}^n, t^n)$.

Writing the Eq. (3.1) for each function $\Phi_h \in Y_h$ we obtain a system of linear equations which enable us to evaluate the values of functions U_h in the points P_{ij}^{n+1} , $i = \overline{1, I-1}$, $j = \overline{1, J-1}$ in terms of their values in the points P_{ij}^n .

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METODA ELEMENTULUI FINIT ÎN ANALIZA OSCILAȚIILOR
SISTEMELOR MAGNETOHIDRODINAMICE BIDIMENSIONALE

(Rezumat)

Se analizează modul în care se poate aplica metoda elementului finit în aproximarea soluției ecuațiilor diferențiale care descriu oscilațiile sistemelor magnetohidrodinamice.

PLANE WAVES IN HEMITROPIC MICROPOLAR CONTINUUM DUE TO MOVING LOADS

BY

VITALI GHEORGHITĂ

1. Introduction. In the last years the author attention was concentrated on the integration of motion equations in the frame of linear micropolar elasticity or thermoelasticity, and for some particular boundary and initial conditions.

Thus, some problems of the propagation of waves and vibrations in thermoelastic hemitropic (without central symmetry) micropolar elastic solid have been investigated by J. P. Nowacki and W. Nowacki in [8], [9].

The steady-state response to moving loads on the free plane boundary of a semi-infinite micropolar elastic medium have been considered by Sengupta and Ghosh [11] and by Kumar and Gogna [5].

The present paper is concerned with the steady-state response to moving loads on the free boundary of a hemitropic micropolar elastic semi-plane. The displacement and microrotation components for the cases of subsonic and supersonic velocities are derived.

2. Basic Equations. The constitutive equations for a hemitropic elastic solid are [6]

$$(2.1) \quad \begin{aligned} t_{ji} &= (\mu + \alpha) \gamma_{ji} + (\mu - \alpha) \gamma_{ij} + \lambda \gamma_{kk} \delta_{ij} + (\chi + \nu) \kappa_{ji} + (\chi - \nu) \kappa_{ij} + k \kappa_{kk} \delta_{ij}, \\ m_{ij} &= (\gamma + \varepsilon) \kappa_{ji} + (\gamma - \varepsilon) \kappa_{ij} + \beta \kappa_{kk} \delta_{ij} + (\chi + \nu) \gamma_{ji} + (\chi - \nu) \gamma_{ij} + k \gamma_{kk} \delta_{ij}, \end{aligned}$$

where the strain tensor γ_{ji} and the curvature-twist tensor κ_{ji} are related to the displacement vector u and the rotation vector φ by

$$(2.2) \quad \gamma_{ji} = u_{i,j} - \varepsilon_{kji} \varphi_k, \quad \kappa_{ji} = \varphi_{i,j}.$$

In (2.1), t_{ji} , m_{ji} represent the stress force tensor and the couple stress tensor, respectively, while λ , μ , α , β , γ , ε , ν , χ , k stand for material constants describing elastic properties of the solid, and ε_{kji} is the alternate tensor.

The equations of motion are [5]

$$(2.3) \quad t_{ji,j} + X_i = \rho \ddot{u}_i, \quad \varepsilon_{ijk} t_{jk} + m_{ji,j} + Y_i = J \ddot{\varphi}_i,$$

where X_i and Y_i are the components of body force and body couple, ρ — density, J — rotational inertia.

Substituting the relations (2.1) into the Eqs. (2.3) and expressing γ_{ji} and κ_{ji} in terms of the displacements u_i and rotations φ_i as determined from (2.2) we arrive to the following coupled equations written in the vector form [7]:

$$(2.4) \quad \begin{aligned} &(\mu + \alpha) \nabla^2 \mathbf{u} + (\lambda + \mu - \alpha) \text{grad div } \mathbf{u} + 2\alpha \text{rot } \boldsymbol{\varphi} + (\chi + \nu) \nabla^2 \boldsymbol{\varphi} + \\ &+ (\chi + k - \nu) \text{grad div } \boldsymbol{\varphi} + \mathbf{X} = \rho \ddot{\mathbf{u}}, \\ &(\gamma + \varepsilon) \nabla^2 \boldsymbol{\varphi} + (\beta + \gamma - \varepsilon) \text{grad div } \boldsymbol{\varphi} + 4\nu \text{rot } \boldsymbol{\varphi} + (\chi + \nu) \nabla^2 \mathbf{u} + \\ &+ (\chi + k - \nu) \text{grad div } \mathbf{u} + \mathbf{Y} = J \ddot{\boldsymbol{\varphi}}. \end{aligned}$$

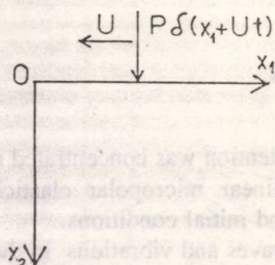


Fig. 1 — A moving load over a micropolar elastic half-plane.

3. Formulation of the Problem. We consider a micropolar, elastic, homogeneous, isotropic and hemitropic body which occupies the semiplane. Let us take Ox_1x_2 as the frame of reference in Cartesian coordinates, the origin O being any point on the boundary $x_2=0$ and x_2 -axis directed normally to the semi-plane represented by $x_2 \geq 0$. Let us assume that a line load moves with a constant speed U over the half-plane in the negative direction of the x_1 -axis (Fig. 1).

An observer moving together with the load at the same speed would see the load as stationary. Thus, if a Galilean transformation

$$(3.1) \quad x'_1 = x_1 + Ut, \quad x'_2 = x_2, \quad t' = t$$

is introduced, then the boundary conditions would be independent of t' since x_1 and t enter only in the combination $x_1 + Ut$.

We confine ourselves to the case when the concentrated load moving over the boundary $x'_2=0$ of semi-plane is of the form

$$(3.2) \quad t_{22}(x'_1, 0) = -P\delta(x'_1), \quad t_{21}(x'_1, 0) = 0, \quad m_{23}(x'_1, 0) = 0,$$

where $\delta(x'_1)$ is the Dirac-delta function.

We are to seek solutions of Eqs. (2.4) subjected to the boundary conditions (3.2).

4. General Solution of the Motion Equations. In the present plane problem, the external loadings, the displacement vector $\mathbf{u}$ and the rotation vector $\boldsymbol{\varphi}$ depend only on the coordinates x_1, x_2 and time t , i.e. $\mathbf{u} = (u_1, u_2, 0)$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, 0)$, $u_i = u_i(x_1, x_2, t)$, $\varphi_i = \varphi_i(x_1, x_2, t)$, $i = 1, 2$; the body forces and the body couples are neglected i.e. $\mathbf{X} = \mathbf{Y} = 0$.

In this case the system (2.4) becomes

$$\begin{aligned}
 & \left[(\mu + \alpha) \nabla^2 - \rho \frac{\partial^2}{\partial t^2} \right] \mathbf{u} + (\lambda + \mu - \alpha) \text{grad div } \mathbf{u} + 2\alpha \text{rot } \boldsymbol{\varphi} + \\
 & + (\chi + \nu) \nabla^2 \boldsymbol{\varphi} + (\chi + k - \nu) \text{grad div } \boldsymbol{\varphi} = 0, \\
 (4.1) \quad & \left[(\gamma + \varepsilon) \nabla^2 - 4\alpha - J \frac{\partial^2}{\partial t^2} \right] \boldsymbol{\varphi} + (\beta + \gamma - \varepsilon) \text{grad div } \boldsymbol{\varphi} + 2\alpha \text{rot } \mathbf{u} + 4\nu \text{rot } \boldsymbol{\varphi} + \\
 & + (\chi + \nu) \nabla^2 \mathbf{u} + (\chi + k - \nu) \text{grad div } \mathbf{u} = 0.
 \end{aligned}$$

To find a solution of the coupled equations (4.1) we decompose the vectors $\mathbf{u}$ and $\boldsymbol{\varphi}$ into potential and solenoidal parts according to the Helmholtz's theorem [4]

$$\begin{aligned}
 (4.2) \quad & \mathbf{u} = \text{grad } \phi + \text{rot } \boldsymbol{\psi}, \quad \text{div } \boldsymbol{\psi} = 0, \\
 & \boldsymbol{\varphi} = \text{grad } \Gamma + \text{rot } \mathbf{H}, \quad \text{div } \mathbf{H} = 0.
 \end{aligned}$$

In the plane case we have: $\phi = \phi(x_1, x_2, t)$, $\Gamma = \Gamma(x_1, x_2, t)$, $\boldsymbol{\psi} = (0, 0, \psi)$, $\mathbf{H} = (0, 0, H)$ with $\psi = \psi(x_1, x_2, t)$, $H = H(x_1, x_2, t)$, and the relations (4.2) can be written in scalar form as

$$\begin{aligned}
 (4.3) \quad & u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_2}, \quad u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_1}, \\
 & \varphi_1 = \frac{\partial \Gamma}{\partial x_1} + \frac{\partial H}{\partial x_2}, \quad \varphi_2 = \frac{\partial \Gamma}{\partial x_2} - \frac{\partial H}{\partial x_1}.
 \end{aligned}$$

Substituting (4.3) into Eqs. (4.1) we conclude that Eqs. (4.1) are satisfied if

$$\begin{aligned}
 (4.4) \quad & \left[(\lambda + 2\mu) \nabla^2 - \rho \frac{\partial^2}{\partial t^2} \right] \phi + (k + 2\chi) \nabla^2 \Gamma = 0, \\
 & \left[(\beta + 2\gamma) \nabla^2 - \rho \frac{\partial^2}{\partial t^2} \right] \Gamma + (k + 2\chi) \nabla^2 \phi = 0, \\
 & \left[(\mu + \alpha) \nabla^2 - \rho \frac{\partial^2}{\partial t^2} \right] \psi = 0, \quad \nabla^2 \psi = 0, \\
 & \left[(\gamma + \varepsilon) \nabla^2 - 4\alpha - J \frac{\partial^2}{\partial t^2} \right] H = 0, \quad \nabla^2 H = 0.
 \end{aligned}$$

After the elimination of ϕ and Γ from the Eqs. (4.4)₁₋₂ we obtain the following independent equations:

$$(4.5) \quad \left[\left[\nabla^2 - v_0^2 - \frac{1}{c_3^2} \frac{\partial^2}{\partial t^2} \right] \left[\nabla^2 - \frac{1}{c_1^2} \frac{\partial^2}{\partial t^2} \right] - v_1^2 \nabla^2 \nabla^2 \right] (\phi, \Gamma) = 0,$$

where

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_3^2 = \frac{\beta + 2\gamma}{J}, \quad v_0^2 = \frac{4\alpha}{\beta + 2\gamma}, \quad v_1^2 = \frac{(k + 2\chi)^2}{(\lambda + 2\mu)(\beta + 2\gamma)}, \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

In view of (3.1), the Eqs. (4.5) take the form

$$(4.6) \quad \left\{ \left[\left[1 - \frac{U^2}{c_3^2} \right] \frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} - v_0^2 \right] \left[\left[1 - \frac{U^2}{c_1^2} \right] \frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} \right] - v_1^2 \left[\frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} \right]^2 \right\} (\phi, \Gamma) = 0.$$

Let us, now, introduce the Mach numbers and the parameters $\beta_i^2, \beta_i'^2$ as

$$(4.7) \quad M_i = \frac{U}{c_i}, \quad \beta_i^2 = 1 - \frac{U^2}{c_i^2} = 1 - m_i^2 \quad \text{if } M_i < 1, \\ \beta_i'^2 = M_i^2 - 1, \quad \text{if } M_i > 1, \quad i = 1, 3.$$

Eqs. (4.6) shall be solved by Fourier integral transform defined by [12]

$$(4.8) \quad \bar{f}(x_2', \xi) = \int_{-\infty}^{+\infty} f(x_1', x_2') e^{i\xi x_1'} dx_1', \\ f(x_1', x_2') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{f}(x_2', \xi) e^{-i\xi x_1'} d\xi.$$

I. *Subsonic case: $M_i < 1$ ($i = 1, 3$).* Using the notations (4.7), the Eqs. (4.6) become

$$(4.9) \quad \left[\left[\beta_3^2 \frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} - v_0^2 \right] \left[\beta_1^2 \frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} \right] - v_1^2 \left[\frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} \right]^2 \right] (\phi, \Gamma) = 0.$$

Taking Fourier transforms of the Eqs. (4.9) we obtain the expressions for ϕ and Γ , satisfying the radial conditions as

$$\phi(x'_1, x'_2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (A(\xi)e^{-\lambda_1 x'_2} + B(\xi)e^{-\lambda_2 x'_2}) e^{-i\xi x'_1} d\xi, \quad (4.10)$$

$$\Gamma(x'_1, x'_2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (C(\xi)e^{-\lambda_1 x'_2} + D(\xi)e^{-\lambda_2 x'_2}) e^{-\lambda_2 \xi x'_1} ds,$$

where

$$\lambda_{1,2}^2 = \frac{1}{2(1-\nu_1^2)} \left[Q_1 \pm \sqrt{Q_1^2 - 4Q_2} \right], \quad Q_1 = (\beta_1^2 + \beta_3^2 - 2\nu_1^2) + \nu_0^2,$$

$$Q_2 = (1 - \nu_1^2) \left[(\beta_1^2 \beta_3^2 - \nu_1^2) \xi^2 + \beta_1^2 \nu_0^2 \right] \xi^2, \quad 1 - \nu_1^2 = \frac{(\lambda + 2\mu)(\beta + 2\gamma) - (k + 2\chi)^2}{(\lambda + 2\mu)(\beta + 2\gamma)} > 0$$

(this inequality derives from energy arguments [7]).

Substituting (4.10) in (4.4) we obtain the following relations between $A(\xi)$, $C(\xi)$ and $B(\xi)$, $D(\xi)$:

$$C(\xi) = \omega_1 A(\xi), \quad D(\xi) = \omega_2 B(\xi), \quad (4.11)$$

where

$$\omega_i = \frac{\beta_1^2 - \lambda_i^2}{\lambda_i^2 - \xi^2} \frac{1}{p}, \quad p = \frac{k + 2\chi}{\lambda + 2\mu}, \quad i = 1, 2.$$

Now we seek a particular solution for the Eqs. (4.4)₃₋₆.

It can be shown easily that a particular solution of these equations which excludes a rigid displacement is $\psi = H = 0$.

Substituting (4.10) in the boundary conditions (3.2) and taking into account (2.1) and (4.11), we get the following system:

$$\begin{aligned} & \left[(\lambda + 2\mu)\lambda_1^2 - \lambda\xi^2 + (k + 2\chi)\lambda_1^2\omega_1 - k\omega_1\xi^2 \right] A(\xi) + \\ & + \left[(\lambda + 2\mu)\lambda_2^2 - \lambda\xi^2 + (k + 2\chi)\lambda_2^2\omega_2 - k\omega_2\xi^2 \right] B(\xi) = -P, \\ & [2\mu\lambda_1 + (\lambda + \chi)\lambda_1\omega_1] A(\xi) + [2\mu\lambda_2 + (\lambda + \chi)\lambda_2\omega_2] B(\xi) = 0. \end{aligned} \quad (4.12)$$

Solving the system (4.12) for $A(\xi)$ and $B(\xi)$ we get

$$a(\xi) = -P \frac{\Delta_1(\xi)}{\Delta(\xi)}, \quad B(\xi) = P \frac{\Delta_2(\xi)}{\Delta(\xi)}, \quad (4.13)$$

where

$$\Delta_1(\xi) = 2\mu\lambda_2 + (\lambda + \chi)\lambda_2\omega_2, \quad \Delta_2(\xi) = 2\mu\lambda_1 + (\lambda + \chi)\lambda_1\omega_1,$$

$$\Delta_3 = [(\lambda + 2\mu)\lambda_1^2 + (k + 2\chi)\lambda_1^2\omega_1 - (\lambda + k\omega_1)\xi^2]\Delta_1(\xi) - \\ - [(\lambda + 2\mu)\lambda_2^2 + (k + 2\chi)\lambda_2^2\omega_2 - (\lambda + k\omega_2)\xi^2]\Delta_2(\xi).$$

Hence, in view of (4.11) we have

$$(4.14) \quad C(\xi) = -P \frac{\omega_1 \Delta_1(\xi)}{\Delta(\xi)}, \quad D(\xi) = P \frac{\omega_1 \Delta_2(\xi)}{\Delta(\xi)}.$$

We can thus easily obtain the components of displacement and microrotation by replacing (4.13) and (4.14), firstly in (4.10) and then in (4.3):

$$(4.15) \quad \begin{aligned} u_1(x'_1, x'_2) &= \frac{iP}{2\pi} \int_{-\infty}^{+\infty} \left(\Delta_1(\xi) e^{-\lambda_1 x'_2} - \Delta_2(\xi) e^{-\lambda_2 x'_2} \right) \frac{\xi}{\Delta(\xi)} e^{-i\xi x'_1} d\xi, \\ u_2(x'_1, x'_2) &= \frac{P}{2\pi} \int_{-\infty}^{+\infty} \left(\lambda_1 \Delta_1(\xi) e^{-\lambda_1 x'_2} - \lambda_2 \Delta_2(\xi) e^{-\lambda_2 x'_2} \right) \frac{1}{\Delta(\xi)} e^{-i\xi x'_1} d\xi, \\ \varphi_1(x'_1, x'_2) &= \frac{iP}{2\pi} \int_{-\infty}^{+\infty} \left(\omega_1 \Delta_1(\xi) e^{-\lambda_1 x'_2} - \omega_2 \Delta_2(\xi) e^{-\lambda_2 x'_2} \right) \frac{\xi}{\Delta(\xi)} e^{-i\xi x'_1} d\xi, \\ \varphi_2(x'_1, x'_2) &= \frac{P}{2\pi} \int_{-\infty}^{+\infty} \left(\lambda_1 \omega_1 \Delta_1(\xi) e^{-\lambda_1 x'_2} - \lambda_2 \omega_2 \Delta_2(\xi) e^{-\lambda_2 x'_2} \right) \frac{1}{\Delta(\xi)} e^{-i\xi x'_1} d\xi. \end{aligned}$$

Making use of the formulae (2.1) and (2.2), we can determine the state of stress and the state of couple-stress in the semi-infinite micropolar medium.

II. *Supersonic case*: $M_i > 1$ ($i=1, 3$). In this case the Eqs. (4.6) with the help of the relations (4.7), take the form

$$(4.16) \quad \left[\left[\beta_3'^2 \frac{\partial^2}{\partial x_1'^2} - \frac{\partial^2}{\partial x_2'^2} + v_0^2 \right] \left[\beta_1'^2 \frac{\partial^2}{\partial x_1'^2} - \frac{\partial^2}{\partial x_2'^2} \right] - \right. \\ \left. - v_1^2 \left[\frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} \right]^2 \right] (\phi, \Gamma) = 0.$$

Using the Fourier transforms (4.8), we obtain the expressions of ϕ and Γ satisfying the radiation conditions, as

$$\phi(x_1', x_2') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (A'(\xi) e^{-i\lambda_1' x_2'} + B'(\xi) e^{-i\lambda_2' x_2'}) e^{-i\xi x_1'} d\xi, \quad (4.17)$$

$$\Gamma(x_1', x_2') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (C'(\xi) e^{-i\lambda_1' x_2'} + D'(\xi) e^{-i\lambda_2' x_2'}) e^{-i\xi x_1'} d\xi,$$

where

$$\lambda_{1,2}'^2 = \frac{1}{2(1-\nu_1'^2)} \left[-Q_1' \pm \sqrt{Q_1'^2 - 4Q_2'} \right], \quad Q_1' = (\beta_1'^2 + \beta_3'^2 + 2\nu_1'^2) \xi^2 - \nu_0'^2,$$

$$Q_2' = (1-\nu_1'^2) \left[(\beta_1'^2 - \nu_1'^2) \xi^2 - \beta_1'^2 \nu_0'^2 \right] \xi^2, \quad C_1'(\xi) = \omega_1' A_1'(\xi), \quad D'(\xi) = \omega_2' B'(\xi),$$

$$\omega_i' = \frac{\beta_1'^2 \xi^2 - \chi_i' \frac{1}{p}}{\xi^2 + \lambda_i'^2} \frac{1}{p}, \quad p = \frac{k+2\chi}{\lambda+2\mu}, \quad i=1, 2.$$

From the boundary conditions (3.2), we get the unknown constants $A'(\xi)$ and $B'(\xi)$ as

$$(4.18) \quad A'(\xi) = P \frac{\Delta_1'(\xi)}{\Delta(\xi)}, \quad B'(\xi) = -P \frac{\Delta_2'(\xi)}{\Delta(\xi)},$$

where

$$\Delta'(\xi) = \left[(\lambda+2\mu) \lambda_1'^2 + (k+2\chi) \lambda_1'^2 + (\lambda+k\omega_1') \xi^2 \right] \Delta_1'(\xi) - \left[(\lambda+2\mu) \lambda_2'^2 + (k+2\chi) \lambda_2'^2 \omega_2' + (\lambda+2\omega_2') \xi^2 \right] \Delta_2'(\xi),$$

$$\Delta_1'(\xi) = 2\mu \lambda_2' + (\lambda+\chi) \lambda_2' \omega_2', \quad \Delta_2'(\xi) = 2\mu \lambda_1' + (\lambda+\chi) \lambda_1' \omega_1'.$$

Using the above relations, we can obtain the displacement and microrotation components:

$$u_1(x_1', x_2') = \frac{-iP}{2\pi} \int_{-\infty}^{+\infty} \left(\Delta_1'(\xi) e^{-i\lambda_1' x_2'} - \Delta_2'(\xi) e^{-i\lambda_2' x_2'} \right) \frac{\xi}{\Delta'(\xi)} e^{-i\xi x_1'} d\xi,$$

$$u_2(x_1', x_2') = \frac{-iP}{2\pi} \int_{-\infty}^{+\infty} \left(\lambda_1' \Delta_1'(\xi) e^{-i\lambda_1' x_2'} - \lambda_2' \Delta_2'(\xi) e^{-i\lambda_2' x_2'} \right) \frac{1}{\Delta'(\xi)} e^{-i\xi x_1'} d\xi,$$

(4.19)

$$\varphi_1(x_1', x_2') = \frac{-iP}{2\pi} \int_{-\infty}^{+\infty} \left(\omega_1' \Delta_1'(\xi) e^{-i\lambda_1' x_2'} - \omega_2' \Delta_2'(\xi) e^{-i\lambda_2' x_2'} \right) \frac{\xi}{\Delta'(\xi)} e^{-i\xi x_1'} d\xi,$$

$$\varphi_2(x'_1, x'_2) = \frac{-iP}{2\pi} \int_{-\infty}^{+\infty} \left(\lambda'_1 \omega'_1 \Delta'_1(\xi) e^{-i\lambda'_1 x'_2} - \lambda'_2 \omega'_2 \Delta'_2(\xi) e^{-i\lambda'_2 x'_2} \right) \frac{1}{\Delta'(\xi)} e^{-i\xi x'_1} d\xi.$$

The *transonic case* $M_1 < 1$, $M_3 > 1$, can be investigated by a similar way.

The case when only the couple stress exists on the boundary of the semi-plane, the boundary conditions in moving coordinates are

$$(4.20) \quad m_{22}(x'_1, 0) = -1 \delta(x'_1), \quad m_{21}(x'_1, 0) = 0, \quad t_{23}(x'_1, 0) = 0.$$

This case may be treated by the same approach as that used in the case of moving stress.

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UNDE PLANE ÎNTR-UN MEDIU CONTINUU MICROPOLAR HEMITROP CAUZATE DE ÎNCĂRCĂRI MOBILE

(Rezumat)

Se studiază starea staționară de răspuns la încărcările mobile aplicate pe frontiera liberă a unui semiplan elastic, micropolar și hemitrop.

Se deduc componentele deplasărilor și microrotațiilor sub formă integrală în cazul vitezelor subsonice și al vitezelor supersonice.

FINITE ELEMENT ANALYSIS FOR THE CASE OF MICROPOLAR ELASTIC MATERIALS (I)

BY

D. MANOLE

1. Introduction. Finite element analysis of elastic bodies has been presented in several papers [3]...[6]. It is the aim of the present paper to establish a finite element analysis for the case of a micropolar elastic material. In Part II will be discussed error estimates in case of regular solution and convergence solution.

2. Preliminary Consideration. Throughout this paper, we employ a rectangular coordinate system x_k and the indicial notation. We consider a regular region Ω of space occupied by a micropolar elastic material, whose Lipschitz boundary is $\partial\Omega$. The basic equations in the linear theory of these bodies are [1]:

i) the *equilibrium equations*

$$(2.1) \quad \begin{aligned} t_{ji,j} + f_i &= 0, \\ m_{ji,j} + \varepsilon_{ijk} t_{jk} + l_i &= 0; \end{aligned}$$

ii) the *constitutive equations*

$$(2.2) \quad \begin{aligned} t_{ij} &= A_{ijkl} \chi_{kl}, \\ m_{ij} &= B_{kl ij} \gamma_{kl} + C_{ijkl} \chi_{kl}; \end{aligned}$$

iii) the *geometrical equations*

$$(2.3) \quad \gamma_{ij} = u_{j,i} + \varepsilon_{jik} \varphi_k, \quad \chi_{ij} = \varphi_{j,i},$$

where

$$(2.4) \quad A_{ijkl} = A_{klij}, \quad C_{ijkl} = C_{klij}.$$

To the system of field equations, we add boundary conditions

$$(2.5) \quad \begin{aligned} u_i &= \bar{u}_i \quad \text{on } S_1, & \varphi_i &= \bar{\varphi}_i \quad \text{on } S_3, \\ t_i &= t_{ij} n_j = \bar{t}_i \quad \text{on } S_2, & m_i &= m_{ji} n_j = \bar{m}_i \quad \text{on } S_4, \end{aligned}$$

where S_r ($r=1, 2, 3, 4$) are parts of $\partial\Omega$ so that

$$S_1 \cup S_2 = S_3 \cup S_4 = \partial\Omega, \quad S_1 \cap S_2 = S_3 \cap S_4 = \emptyset.$$

In the above equations, we have used the following notation: t_{ij} — components of the stress tensor; m_{ij} — components of the couple stress tensor; u_i — components of the displacement vector; φ_i — components of the microrotation vector; γ_{ij} , χ_{ij} — components of deformation; ε_{ijk} — alternating symbol; f_i — components of body force; l_i — components of body couple; n_i — components of the unit outward normal to $\partial\Omega$; the comma denotes partial derivation with respect to the variable x_i ; A_{ijkl} , B_{ijkl} , C_{ijkl} are functions characterizing the respective material and $\tilde{u}_i$, $\tilde{\varphi}_i$, $\tilde{t}_i$, $\tilde{m}_i$ — are prescribed functions.

Hence forth we denote by $H^k(\Omega)$ the Sobolev space $W^k, 2(\Omega)$,

$$(f, g)_0 = \int_{\Omega} fg \, d\Omega, \quad (u, v)_{0,s} = \int_{\Omega} u v \, ds;$$

$\|u\|_k$ is the norm in $H^k(\Omega)$, $k=0, 1, \dots$, generated by all derivatives of the k -th order. Let be

$$\begin{aligned} u &= (u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3), \quad v = (v_1, v_2, v_3, \omega_1, \omega_2, \omega_3), \\ V &= \{v \in (H^1(\Omega))^6 \mid v=0 \text{ on } S_1 \cup S_3\} \Leftrightarrow \{v_i \in H^1(\Omega) \mid v_i=0 \text{ on } S_1, \\ &\quad \omega_i \in H^1(\Omega) \mid \omega_i=0 \text{ on } S_3; i=1, 2, 3\}. \end{aligned}$$

3. Derivation of a Variational Principle for Micropolar Elastostatics. We introduce the notation

$$(3.1) \quad A(\gamma_{ij}, \chi_{ij}) = A_{ijkl} \gamma_{ij} + 2B_{ijkl} \gamma_{ij} \chi_{kl} + C_{ijkl} \chi_{ij} \chi_{kl},$$

$$(3.2) \quad a(u, v) = \int_{\Omega} (t_{ij}(u) \gamma_{ij}(v) + m_{ij}(u) \chi_{ij}(v)) \, d\Omega,$$

$$(3.3) \quad F(v) = \int_{\Omega} (f_i v_i + l_i \omega_i) \, d\Omega + \int_{S_2} \tilde{t}_i v_i \, ds + \int_{S_4} \tilde{m}_i \omega_i \, ds$$

and suppose that

$$(3.4) \quad A(\gamma_{ij}, \chi_{ij}) \geq c \sum_{i,j=1}^3 (\gamma_{ij}^2 + \chi_{ij}^2), \quad c > 0.$$

Multiplying (2.2)₁ by $\gamma_{ij}(v)$ and (2.2)₂ by $\chi_{ij}(v)$, summing up, integrating over Ω , applying the divergence theorem and taking account of (2.1), (2.3)–(2.5), (3.2) and (3.3), it results that

$$(3.5) \quad a(u, v) = F(v), \quad \forall v \in V \quad (\text{the variational equality}).$$

The linearity of $F(\cdot)$ and bilinearity of $a(\cdot, \cdot)$ on the space V are obvious.

Using relation (2.4), from (3.2), we have

$$(3.6) \quad a(u, v) = a(v, u), \quad \forall u, v \in V.$$

From (3.1), (3.2) and (3.4), we obtain

$$(3.7) \quad a(u, v) = \int_{\Omega} A(\gamma_{ij}, \chi_{ij}) \, d\Omega$$

and

$$(3.8) \quad a(v, v) \geq c \int_{\Omega} \sum_{i,j=1}^3 (\gamma_{ij}^2(v) + \chi_{ij}^2(v)) d\Omega.$$

From (3.8), we have (see [1])

$$(3.9) \quad a(v, v) \geq c_1 \|v\|_{1,\Omega}^2, \quad c_1 > 0, \quad \forall v \in V \text{ (the } V\text{-ellipticity condition).}$$

We suppose that

$$(3.10) \quad |a(u, v)| \leq c_2 \|u\|_{1,\Omega} \|v\|_{1,\Omega}, \quad c_2 > 0, \quad \forall u, v \in V.$$

The continuity of $F(v)$ follows immediately from the Cauchy-Schwartz inequality.

Thus, by Theorem 2.1 (see [2]), the variational problem:

Find $u \in V$ such that

$$(3.11) \quad a(u, v) = F(v), \quad \forall v \in V$$

has one and only one solution.

Since $F(\cdot)$ is a continuous linear form, there exists by the Riesz representation theorem a unique element $u = u_F \in V$ such that

$$(3.12) \quad F(v) = a(u_F, v), \quad \forall v \in V.$$

From (3.6), (3.9) and (3.10) it results that (see [2]) the problem (3.11) is equivalent to the problem: Find $u \in V$ such that

$$(3.13) \quad J(u) = \inf_{v \in V} J(v),$$

where

$$(3.14) \quad J(v) = \frac{1}{2} a(v, v) - F(v).$$

4. The Basic Idea of Finite Element Method. The finite element method in its simplest setting is a Galerkin method characterized by three basic aspects in the construction of the space V_n :

- i) a triangulation T_n is established over the set $\bar{\Omega}$;
- ii) the functions $v_n \in V_n$ are piecewise polynomials;
- iii) there exists a basis in the space V_n whose functions have small supports.

The so-called discretization parameter h will be specified later. First of all let V_n be an arbitrary finite dimensional subspace of V . Then the Galerkin method for approximating the solution of the problem (3.11) consists in finding $u_n \in V_n$ such that

$$(4.1) \quad a(u_n, v_n) = F(v_n), \quad \forall v_n \in V_n.$$

Applying Theorem 2.17 [2], we infer that such a problem has one and only one solution u_n , which we shall call the discrete solution.

The discrete solution is also characterized by the property (see Theorem 2.12 [2]):

$$(4.2) \quad J(u_n) = \inf_{v_n \in V_n} J(v_n),$$

where J is given by (3.14). The alternative definition of the discrete solution is known as the Ritz method.

Note that

$$(4.3) \quad J(u) \leq J(u_n),$$

since $V_n \subset V$. From (3.11) and (4.1) we get the following orthogonality relation

$$(4.4) \quad a(u - u_n, v_n) = 0, \quad \forall v_n \in V_n,$$

which says that the error $u - u_n$ is orthogonal to V_n with respect to the scalar product $a(\cdot, \cdot)$.

Let $\{v^i\}_{i=1}^m$ be a basis in V_n . We shall look for the discrete solution u_n as a linear combination of the basis functions

$$(4.5) \quad u_n = \sum_{j=1}^m c_j v^j.$$

Then by (4.1) it is true that

$$(4.6) \quad a\left(\sum_{j=1}^m c_j v^j, v^i\right) = F(v^i), \quad \forall i = 1, \dots, m \quad (= \dim V_n).$$

This finally leads to the following system of algebraic equations

$$(4.7) \quad \sum_{j=1}^m a(v^j, v^i) c_j = F(v^i), \quad i = 1, \dots, m,$$

for the unknowns $c_1, \dots, c_m$.

The matrix $A = (a(v^j, v^i))_{i,j=1}^m$ and the vector $(F(v^i))_{i=1}^m$ are often called the stiffness matrix and the load vector, respectively. A is nonsingular (see [2], p. 25).

Continuing, we subdivide the set $\bar{\Omega}$ into a finite number of subsets K (called elements) in such a way that the following properties hold:

- $\bar{\Omega} = \bigcup_{K \in T_n} K$;
- for each $K \in T_n$, the set K is closed and its interior K^0 is non-empty;
- for each distinct $K_1, K_2 \in T_n$ one has $K_1^0 \cap K_2^0 = \emptyset$;
- for each $K \in T_n$, the boundary ∂K is Lipschitzian.

The discretization (triangulation) parameter h is the maximum diameter of all $K \in T_n$.

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ANALIZA ELEMENTULUI FINIT PENTRU CAZUL MATERIALULUI ELASTIC MICROPOLAR (I)

(Rezumat)

Se pune în evidență metoda elementului finit în cazul elasticității liniare micropolare. Probleme legate de evaluarea erorii și convergența soluției vor fi date în partea a II-a.

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ANALYSIS OF THE THERMAL STABILITY OF ELASTIC POLYMERS

(Received)

The thermal stability of elastic polymers is a subject of increasing importance in the study of the kinetics of polymer degradation. The present paper is devoted to the study of the thermal stability of elastic polymers, and to the determination of the rate constants of the various steps in the degradation process.

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POLARIZATION GRADIENT IN A GRADE CONSISTENT MICROPOLAR THEORY OF PIEZOELECTRICITY

BY

D. VIERU and SILVIA GABRIELA CIUMAȘU

1. Introduction. The linear theory of piezoelectricity was given by W. Voigt [10]. Numerous anomalies in behaviour of some crystals, both in piezoelectric and mechanical effects, prove the necessity of constructing an appropriate theory. A linear theory of thermopiezoelectricity in micropolar medium was given by W. Nowacki [6]. By inclusion of the polarization gradient in the stored energy function of elastic dielectrics, the classical theory of piezoelectricity was extended by R. D. Mindlin [5]. Brulin and Hjalmars [1], [2] derived a theory of micropolar continua, where the second-order displacement gradient is added to the classic set of independent constitutive variables.

In this paper we derive a linear theory of piezoelectricity for non-simple micropolar medium, where the polarization gradient is added to the set of independent constitutive variables. We deduce the governing equation of the linear theory for anisotropic solid by means of a variational principle.

2. Basic Equations. We consider the motion of the continuum in a fixed system rectangular Cartesian axes OX_i ($i=1, 2, 3$). The body occupies a bounded region V with Lipschitz boundary S which consists of a finite number of smooth surfaces. Denote by γ_α the crossing of two jointed smooth surfaces $\Gamma = \bigcup \gamma_\alpha$. We denote by u_i the components of the displacement vector and by φ_i^α the components of the microrotation vector.

The extension proposed in the classical theory is obtained by inclusion of the second-order displacement gradient and the polarization gradient in the stored energy function of micropolar piezoelectric materials.

Let

$$(2.1) \quad K = \frac{1}{2} (\rho v_i v_i + I_{ij} v_i v_j)$$

be the kinetic energy density, where $v_i = \dot{u}_i$, $v_i = \dot{\varphi}_i$, ρ —the density in the reference

configuration, I_{ij} - the coefficients of microinertia. We define the electric enthalpy density as

$$(2.2) \quad H = W(e_{ij}, \chi_{ij}, \chi_{ijk}, P_i, P_{j,i}) - \frac{1}{2} \epsilon_0 \phi_{,i} \phi_{,i} + \phi_{,i} P_i,$$

where P_i is the polarization, ϵ_0 is the permittivity of vacuum, ϕ is the electric potential of the Maxwell self-field,

$$(2.3) \quad E_i = -\phi_{,i}$$

and e_{ij} , χ_{ij} , χ_{ijk} are the strain displacement tensors,

$$(2.4) \quad e_{ij} = u_{j,i} + \epsilon_{ijk} \varphi_k, \quad \chi_{ij} = \varphi_{j,i}, \quad \chi_{ijk} = u_{k,ij}.$$

Consider two arbitrary instants t_0 and t_1 . Then, for all admissible variation motions of the body, the form of a generalized Hamilton's principle may be stated as follows

$$(2.5) \quad \delta \int_{t_0}^{t_1} dt \int_V (K - H) dv + \int_{t_0}^{t_1} dt \int_V (F_i \delta u_i + G_i \delta \varphi_i + E_i^0 \delta P_i - f \delta \phi) dv + \\ + \int_{t_0}^{t_1} dt \int_S (t_i \delta u_i + m_i \delta \varphi_i + m_{ki} \delta u_{i,k} - f_1 \delta \phi) dA = 0,$$

where: F_i - components of the body force vector per unit volume in the reference configuration; G_i - components of the body couple vector; t_i - a part of the stress vector associated with the surface S ; m_i - components of the couple stress vector; m_{ki} - the dipolar surface force; f - volume density of free charge; E_i^0 - electric field; f_1 - surface density of free charge. For independent variations of u_i , $u_{i,j}$, φ_i , P_i and ϕ ,

$$(2.6) \quad \delta H = \frac{\partial W}{\partial e_{ij}} \delta e_{ij} + \frac{\partial W}{\partial \chi_{ij}} \delta \chi_{ij} + \frac{\partial W}{\partial \chi_{ijk}} \delta \chi_{ijk} + \frac{\partial W}{\partial P_i} \delta P_i + \\ + \frac{\partial W}{\partial P_{j,i}} \delta P_{j,i} - \epsilon_0 \phi_{,i} \delta \phi_{,i} + \phi_{,i} \delta P_i + P_i \delta \phi_{,i}.$$

We introduce the notations

$$(2.7) \quad \tau_{ij} = \frac{\partial W}{\partial e_{ij}}, \quad \mu_{ij} = \frac{\partial W}{\partial \chi_{ij}}, \quad \mu_{ijk} = \frac{\partial W}{\partial \chi_{ijk}}, \\ \pi_i = -\frac{\partial W}{\partial P_i}, \quad \pi_{ij} = \frac{\partial W}{\partial P_{j,i}},$$

where: τ_{ij} - components of the stress tensor; μ_{ij} - components of the couple-stress tensor; μ_{ijk} - the hyperstress tensor; π_i - the effective local electric force; π_{ij} - components of the dipolar electric tensor.

Using the relations (2.4), (2.6), (2.7), we obtain

$$\begin{aligned}
 \delta H = & \tau_{ij} \delta u_{j,i} + \tau_{ij} \varepsilon_{ijk} \delta \varphi_k + \mu_{ij} \delta \varphi_{j,i} + \mu_{ijk} \delta u_{k,ij} - \\
 & - \pi_i \delta P_i + \pi_{ij} \delta P_{j,i} - \varepsilon_0 \phi_{,i} \delta \phi_{,i} + \phi_{,i} \delta P_i + P_i \delta \phi_{,i} = \\
 (2.8) \quad & = (\tau_{ji} \delta u_{i,j})_{,j} - \tau_{ji,j} \delta u_i - \tau_{jk} \varepsilon_{jki} \delta \varphi_i + (\mu_{ji} \delta \varphi_{i,j})_{,j} - \mu_{ji,j} \delta \varphi_i + \\
 & + (\mu_{kji} \delta u_{i,k})_{,j} - (\mu_{kji,j} \delta u_i)_{,k} + \mu_{kji,jk} \delta u_i - \pi_i \delta \varphi_i + \\
 & + (\pi_{ji} \delta P_{i,j})_{,j} - \pi_{ji,j} \delta P_i - (\varepsilon_0 \phi_{,i} \delta \phi_{,i})_{,i} + \varepsilon_0 \phi_{,ii} \delta \phi + \phi_{,i} \delta P_i + \\
 & + (P_i \delta \phi_{,i})_{,i} - P_{i,i} \delta \phi.
 \end{aligned}$$

On the other hand

$$(2.9) \quad \delta \int_{t_0}^{t_1} dt \int_V \left[\frac{1}{2} (\rho \dot{u}_i \dot{u}_i + I_{ij} \dot{\varphi}_i \dot{\varphi}_j) \right] dv = - \int_{t_0}^{t_1} dt \int_V (\rho \ddot{u}_i \delta u_i + I_{ij} \ddot{\varphi}_j \delta \varphi_i) dv.$$

From (2.5), (2.8), (2.9) it follows that

$$\begin{aligned}
 (2.10) \quad & \int_{t_0}^{t_1} dt \int_V \left\{ \left[(\tau_{ji} - \mu_{kji,k})_{,j} + F_i - \rho \ddot{u}_i \right] \delta u_i + [\mu_{ji,j} + \varepsilon_{jki} \tau_{jk} + G_i - I_{ij} \ddot{\varphi}_j] \delta \varphi_i + \right. \\
 & + \left[\pi_i + \pi_{ji,j} - \phi_{,i} + E_i^0 \right] \delta P_i + [P_{i,i} - \varepsilon_0 \phi_{,ii} - f] \delta \phi \Big\} dv + \\
 & + \int_{t_0}^{t_1} dt \int_S \left\{ [t_i - (\tau_{ji} - \mu_{kji,k}) n_j] \delta u_i + [m_i - \mu_{ji} n_j] \delta \varphi_i + \right. \\
 & + \left. [m_{ki} - \mu_{jki} n_j] \delta u_{i,k} - [\pi_{ji} n_j] \delta P_i + [(\varepsilon_0 < \phi_{,i} > P_i) n_i - f_1] \delta \phi \right\} dA = 0,
 \end{aligned}$$

where: n_i — components of the outward unit normal at S ; $<f>$ — the difference of limits from both sides of S .

Since the variations δu_i , $\delta \varphi_i$, δP_i , $\delta \phi$, $\delta u_{i,k}$ are arbitrary, (2.10) leads to the governing equations of the motion and of the electric field

$$(2.11) \quad (\tau_{ji} - \mu_{kji,k})_{,j} + F_i = \rho \ddot{u}_i,$$

$$\mu_{ji,j} + \varepsilon_{ijk} \tau_{jk} + G_i = I_{ij} \ddot{\varphi}_j, \quad \text{in } V;$$

$$(2.12) \quad \pi_{ji,j} + \pi_i - \phi_{,i} + E_i^0 = 0,$$

$$P_{i,i} - \varepsilon_0 \phi_{,ii} = f, \quad \text{in } V,$$

together with the boundary conditions

$$\begin{aligned}
 t_i &= (\tau_{ji} - \mu_{kji, k}) n_j, \quad m_i = \mu_{ji} n_j, \\
 m_{ki} &= \mu_{jki} n_j, \quad p_i = \pi_{ji} n_j = 0, \\
 (\varepsilon_0 < \phi, i > -P_i) n_i &= f_1.
 \end{aligned}
 \tag{2.13}$$

If we introduce the notation

$$t_{ji} = \tau_{ji} - \mu_{kji, k}, \tag{2.14}$$

the Eq. (2.11)₁ and the boundary condition (2.13)₁ can be written

$$t_{ji, j} + F_i = \rho \ddot{u}_i, \tag{2.15}$$

$$t_i = t_{ji} n_j. \tag{2.16}$$

In the linear theory with the assumption that the initial body is free from stress and hyperstress, the energy density of deformation and polarization W (2.2) is taken as

$$\begin{aligned}
 W &= \frac{1}{2} A_{ijkl} e_{ij} e_{kl} + A'_{ijkl} e_{ij} \chi_{kl} + A_{ijklm} e_{ij} \chi_{klm} - A_{ijk} e_{ij} P_k - G_{ijkl} e_{ij} P_{l, k} + \\
 &+ \frac{1}{2} B_{ijkl} \chi_{ij} \chi_{kl} + B_{ijklm} \chi_{ij} \chi_{klm} - B_{ijk} \chi_{ij} P_k - G'_{ijkl} \chi_{ij} P_{l, k} + \\
 &+ \frac{1}{2} D_{ijklmn} \chi_{ijk} \chi_{lmn} - B'_{ijkl} \chi_{ijk} P_l - G_{ijklm} \chi_{ijk} P_{m, l} - \\
 &- \frac{1}{2} A_{ij} P_i P_j - A'_{ijk} P_i P_{k, j} - \frac{1}{2} D_{ijkl} P_{j, i} P_{l, k} + c_{ij} P_{j, i}.
 \end{aligned}
 \tag{2.17}$$

The significance of the linear term $c_{ij} P_{j, i}$ in (2.17) is considered in [5], [8].

Then, from (2.7), (2.17) we obtain

$$\begin{aligned}
 \tau_{ij} &= A_{ijkl} e_{kl} + A'_{ijkl} \chi_{kl} + A_{ijklm} \chi_{klm} - A_{ijk} P_k - G_{ijkl} P_{l, k}, \\
 \mu_{ij} &= A'_{kl ij} e_{kl} + B_{ijkl} \chi_{kl} + B_{ijklm} \chi_{klm} - B_{ijk} P_k - G'_{ijkl} P_{l, k}, \\
 \mu_{ijk} &= A_{lmij k} e_{lm} + B_{lmij k} \chi_{lm} + D_{ijklmn} \chi_{lmn} - B'_{ijkl} P_l - G_{ijklm} P_{m, l}, \\
 \pi_i &= A_{kji} e_{kj} + B_{kji} \chi_{kj} + B'_{klji} \chi_{klj} + A_{ij} P_j + A'_{ijk} P_{k, j}, \\
 \pi_{ij} &= -G_{klji} e_{kl} - G'_{klji} \chi_{kl} - G_{klmij} \chi_{klm} - A'_{kij} P_k - D_{ijkl} P_{l, k} + c_{ij}.
 \end{aligned}
 \tag{2.18}$$

The coefficients in (2.18) enjoy the following symmetries:

$$\begin{aligned}
 A_{ij} &= A_{ji}, \quad A_{ijk} = A_{kij}, \quad A'_{ijk} = A'_{kij}, \quad A_{ijkl} = A_{klij}, \\
 A'_{ijkl} &= A'_{kl ij}, \quad A_{ijklm} = A_{klmij} = A_{ijlkm}, \quad B_{ijk} = B_{kij}, \\
 B_{ijkl} &= B_{kl ij}, \quad B'_{ijkl} = B'_{jikl} = B'_{lij k}, \quad B_{ijklm} = B_{ijlkm} = B_{klmij},
 \end{aligned}
 \tag{2.19}$$

$$D_{ijklmn} = D_{lmnij k} = D_{jiklmn}, \quad D_{ijkl} = D_{klij}, \quad G_{ijkl} = G_{klij},$$

$$G_{ijkl}^{\prime} = G_{klij}^{\prime}, \quad G_{ijklm} = G_{lmijk} = G_{jiklm}.$$

In the last integral of (2.5) $u_{i,k}$ is not independent of u_i on S ; only the normal component $Du_i = u_{i,r} n_r$ is independent. We separate later

$$(2.20) \quad m_{ki} \delta u_{i,k} = \mu_{sjl} n_s n_j \delta u_{i,r} n_r + [\mu_{pij} n_p n_r - \mu_{pri} n_p n_j]_{,r} n_j \delta u_i + \\ + \varepsilon_{mrl} n_m (\varepsilon_{sjl} n_s n_p \mu_{pji} \delta u_i)_{,r}.$$

From Stokes's theorem we have

$$(2.21) \quad \int_S (t_i \delta u_i + m_{ki} \delta u_{i,k}) dA = \\ = \int_S (A_i \delta u_i + B_i D(\delta u_i)) dA + \int_{\Gamma} C_i \delta u_i ds,$$

where

$$(2.22) \quad A_i = [\tau_{ji} - \mu_{kji,k} + (\mu_{pji} n_p n_r - \mu_{pri} n_p n_j)_{,r}] n_j, \\ B_i = \mu_{sjl} n_s n_j, \\ C_i = \langle \varepsilon_{sjl} n_s n_p \mu_{pji} s_l \rangle,$$

where $\langle h \rangle$ denotes the difference of limits from both sides of γ_α and s_i represents the unit tangent vector to γ_α .

Thus the basic equations in this theory are (2.3), (2.4), (2.11), (2.12), (2.18).

Let us consider the subsets $\Sigma_i (i = \bar{1}, 8)$ of S so that

$$\Sigma_1 \cup \Sigma_2 \cup \Gamma = \Sigma_3 \cup \Sigma_4 \cup \Gamma = \Sigma_5 \cup \Sigma_6 \cup \Gamma = \Sigma_7 \cup \Sigma_8 \cup \Gamma = S;$$

$$\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Sigma_5 \cap \Sigma_6 = \Sigma_7 \cap \Sigma_8 = \emptyset$$

and the following boundary conditions

$$(2.23) \quad u_i = \tilde{u}_i \text{ on } \Sigma_1 \times I, \quad A_i = \tilde{A}_i \text{ on } \Sigma_2 \times I, \\ \varphi_i = \tilde{\varphi}_i \text{ on } \Sigma_3 \times I, \quad m_i = \tilde{m}_i \text{ on } \Sigma_4 \times I, \\ Du_i = \tilde{g}_i \text{ on } \Sigma_5 \times I, \quad B_i = \tilde{B}_i \text{ on } \Sigma_6 \times I, \\ \phi = \tilde{\phi} \text{ on } \Sigma_7 \times I, \quad (\varepsilon_0 < \phi_{,i} > P_i) n_i = f_1 \text{ on } \Sigma_8 \times I, \\ p_i = \pi_{ji} n_j = 0 \text{ on } S \times I, \quad P_i = \tilde{P}_i \text{ on } S \times I, \\ C_i = \tilde{C}_i \text{ on } \Gamma \times I,$$

where $I=[0, t_1]$ and $\tilde{u}_i, \tilde{A}_i, \tilde{\varphi}_i, \tilde{m}_i, \tilde{g}_i, \tilde{B}_i, \tilde{\phi}, \tilde{P}_i, \tilde{C}_i$ are prescribed functions and t_1 is an instant that may be infinite.

To the system of field equations (2.3), (2.4), (2.11), (2.12) we adjoin the boundary conditions (2.23) and the initial conditions

$$(2.24) \quad \begin{aligned} u_i(x, 0) &= \tilde{a}_i(x), & \dot{u}_i(x, 0) &= \tilde{b}_i(x), \\ \varphi_i(x, 0) &= \tilde{c}_i(x), & \dot{\varphi}_i(x, 0) &= \tilde{d}_i(x), \end{aligned}$$

where $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i, \tilde{d}_i$ are prescribed functions.

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GRADIENTUL POLARIZĂRII ÎN TEORIA PIEZOELECTRICITĂȚII MATERIALELOR NESIMPLE

(Rezumat)

Se determină ecuațiile de bază în teoria liniară a materialelor piezoelectrice nesimple, introducând printre variabilele constitutive clasice și gradientul polarizării.

THE DIRECT STUDY OF SOME CLASSICAL PROBLEMS OF THEORETICAL MECHANICS

BY

DANIEL CONDURACHE and ALFRED BRAIER

0. Introduction. The aim of this paper is to show that exact solutions of some classical problems of theoretical mechanics, not yet signaled, reduce to the direct integration of the following types of vectorial differential equations (Cauchy's problems):

$$(1) \quad \dot{\mathbf{y}} + \mathbf{b} \times \mathbf{y} = \mathbf{f}, \quad \mathbf{y}(0) = \mathbf{y}_0;$$

and

$$(2) \quad \ddot{\mathbf{y}} + \mathbf{b} \times \dot{\mathbf{y}} + \lambda \mathbf{y} = \mathbf{0}, \quad \lambda \in \mathbb{R},$$

$$(3) \quad \ddot{\mathbf{y}} + 2\mathbf{b} \times \dot{\mathbf{y}} + \mathbf{b} \times (\mathbf{b} \times \mathbf{y}) = \mathbf{f},$$

with the initial conditions

$$(4) \quad \mathbf{y}(0) = \mathbf{y}_0, \quad \dot{\mathbf{y}}(0) = \mathbf{y}_0^*.$$

For particular vectors $\mathbf{b}$ and $\mathbf{f}$, the full expression of $\mathbf{y}(t)$ can be found.

In what follows we consider only a few interesting problems.

1. Motion of a Charged Particle in a Purely Magnetic Field with Fixed Direction. The motion is described by [1], [2]

$$(5) \quad m \frac{d\mathbf{v}}{dt} = q \mathbf{v} \times \mathbf{B}, \quad \mathbf{v}(0) = \mathbf{v}_0,$$

where: m — mass, q — electric charge, $\mathbf{v}$ — velocity, $\mathbf{B}$ — magnetic induction, supposed to be

$$(6) \quad \mathbf{B} = B(t) \mathbf{u}, \quad |\mathbf{u}| = 1, \quad \mathbf{u} = \text{const.},$$

with $B(t)$ slowly variable.

Let write Eq. (5) as

$$(7) \quad \dot{\mathbf{v}} + \omega \times \mathbf{v} = \mathbf{0} \quad \text{with} \quad \omega = \frac{q}{m} \mathbf{B} \left[= \frac{qB(t)}{m} \mathbf{u} \right],$$

where $\omega = 2\omega_L$ is the double of Larmor frequency. The motion equation is of type (1), with $\mathbf{f} = \mathbf{0}$.

The vectorial product with $\mathbf{u}$ of (7) gives

$$(8) \quad \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) + \omega(t) \mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}.$$

Let derivate (7) and substitute the obtained $\dot{\mathbf{v}}$ by its expression from the same Eq. (7), getting $\ddot{\mathbf{v}} + \dot{\omega}(\mathbf{u} \times \mathbf{v}) + \omega^2 \mathbf{v} - \omega \mathbf{u}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{0}$. Now perform here again a vectorial product with $\mathbf{u}$ and eliminate the new appeared vectorial product $\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$ by means of (8).

We get the vectorial linear differential equation

$$(9) \quad \frac{d^2}{dt^2}(\mathbf{u} \times \mathbf{v}) - \frac{\dot{\omega}}{\omega} \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) + \omega^2(\mathbf{u} \times \mathbf{v}) = \mathbf{0},$$

with the immediate solution

$$(10) \quad \mathbf{u} \times \mathbf{v} = \mathbf{C}_1 \cos \theta(t) + \mathbf{C}_2 \sin \theta(t), \quad \text{where} \quad \theta(t) = \frac{q}{m} \int_0^t B(t) dt.$$

Returning with (10)₁ to (7)₁ we find $\dot{\mathbf{v}} = -\mathbf{B} \mathbf{u} \times \mathbf{v}$ and a simple integration leads to

$$(11) \quad \mathbf{v} = -\mathbf{C}_1 \sin \theta(t) + \mathbf{C}_2 \cos \theta(t) + \mathbf{C}_3.$$

In order to determine the integration constants, it is convenient to use (10), (11) and the derivative of (10)₁ introduced in (8), leading to

$$(12) \quad \mathbf{C}_1 = \mathbf{u} \times \mathbf{v}_0, \quad \mathbf{C}_2 = -\mathbf{u} \times (\mathbf{u} \times \mathbf{v}_0), \quad \mathbf{C}_3 = (\mathbf{u} \cdot \mathbf{v}_0) \mathbf{u}.$$

Finally, the solution to (5) is

$$(13) \quad \mathbf{v} = \mathbf{v}_0 - \mathbf{u} \times \{ \mathbf{v}_0 \sin \theta(t) - (\mathbf{u} \times \mathbf{v}) [1 - \cos \theta(t)] \},$$

where $\theta(t)$ is given by (10)₂.

Since $\mathbf{v} = \dot{\mathbf{r}}$, with $\mathbf{r}(0) = \mathbf{r}_0$ we have also the not signaled vectorial form of the law of motion, namely

$$(14) \quad \mathbf{r} = \mathbf{r}_0 + (\mathbf{u} \cdot \mathbf{v}_0) \mathbf{u} t - \mathbf{u} \times \left[\mathbf{v}_0 \int_0^t \sin \theta(\tau) d\tau + (\mathbf{u} \times \mathbf{v}_0) \int_0^t \cos \theta(\tau) d\tau \right].$$

Remarks. 1. The result (13) shows that the velocity hodograph is a circle in a plane perpendicular on the magnetic field fixed direction. The axis of the cone described by vector $\mathbf{v}$ ($|\mathbf{v}| = \text{const.}$) is parallel to $\mathbf{u}$, the velocity accomplishing an irregular precession around the magnetic lines, with the angular velocity $\omega^* = -\omega$.

2. If the magnetic field is uniform, $\mathbf{B} = \text{const.}$ and either again from (7)₁ with $\omega = \text{const.}$, or from (13),

$$(15) \quad \mathbf{v} = \mathbf{v}_0 - \mathbf{u} \times [\mathbf{v}_0 \sin \omega t - \mathbf{u} \times \mathbf{v}_0 (1 - \cos \omega t)].$$

2. Motion of a Charged Particle in a Uniform Magnetic and Electric Field.

If $\mathbf{E} = \text{const.}$ and $\mathbf{B} = \text{const.}$ are the electric and the magnetic vectors, the equation of motion is [2], [3]

$$(16) \quad m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad \mathbf{v}(0) = \mathbf{v}_0,$$

or

$$(17) \quad \dot{\mathbf{v}} + \omega \times \mathbf{v} = \alpha, \quad \text{with} \quad \omega = \frac{q}{m} \mathbf{B}, \quad \alpha = \frac{q}{m} \mathbf{E},$$

both constant vectors.

Denote $\dot{\mathbf{v}} = \mathbf{a}$ and derivate (17)₁ to obtain the Cauchy's problem

$$(18) \quad \dot{\mathbf{a}} + \omega \times \mathbf{a} = \mathbf{0}, \quad \mathbf{a}(0) = \alpha - \omega \times \mathbf{v}_0$$

for an equation like (7)₁, with the previous solution (15), becoming now

$$(19) \quad \mathbf{a} = \alpha - \omega \times \mathbf{v}_0 - \mathbf{u} \times (\alpha - \omega \times \mathbf{v}_0) \sin \omega t + \mathbf{u} \times [\mathbf{u} \times (\alpha - \omega \times \mathbf{v}_0)] (1 - \cos \omega t).$$

The function $\mathbf{v}(t)$ results from $\mathbf{v} = \int \mathbf{a} dt + \mathbf{K}$, with the constant $\mathbf{K}$ determined taking into account (16)₂. We get as general solution of (17)

$$(20) \quad \mathbf{v} = \frac{1}{\omega} \alpha \times \mathbf{u} + (\mathbf{v}_0 \cdot \mathbf{u}) \mathbf{u} + \frac{1}{\omega} (\alpha + \mathbf{v}_0 \times \omega) \sin \omega t + \\ + \frac{1}{\omega} \mathbf{u} \times (\alpha + \mathbf{v}_0 \times \omega) \cos \omega t + \frac{1}{\omega} (\alpha \cdot \mathbf{u}) (\omega t - \sin \omega t) \mathbf{u}.$$

Returning to (17)₂

$$(21) \quad \mathbf{v} = \frac{1}{B^2} \mathbf{E} \times \mathbf{B} + \frac{1}{B^2} (\mathbf{v}_0 \cdot \mathbf{B}) \mathbf{B} + \frac{1}{B} (\mathbf{E} + \mathbf{v}_0 \times \mathbf{B}) \sin \omega t + \\ + \frac{1}{B^2} \mathbf{B} \times (\mathbf{E} + \mathbf{v}_0 \times \mathbf{B}) \cos \omega t + \frac{1}{B^3} (\mathbf{E} \cdot \mathbf{B}) (\omega t - \sin \omega t) \mathbf{B};$$

we have a result not mentioned in such a compact form. The first term is obviously the drift velocity.

If the uniform fields are crossed ($\mathbf{E} \cdot \mathbf{B} = 0$), then

$$(22) \quad \mathbf{v} = \frac{1}{B^2} \mathbf{E} \times \mathbf{B} + \frac{1}{B^2} (\mathbf{v}_0 \cdot \mathbf{B}) \mathbf{B} + \frac{1}{B} (\mathbf{E} + \mathbf{v}_0 \times \mathbf{B}) \sin \omega t + \frac{1}{B^2} \mathbf{B} \times (\mathbf{E} + \mathbf{v}_0 \times \mathbf{B}) \cos \omega t,$$

expression obtained currently by exploiting the linearity of the differential equation [2], [4]. The motion law results immediately.

Remarks. 1. When $\mathbf{E} \cdot \mathbf{B} \neq 0$, the velocity hodograph is a helix with variable step, enveloped on an elliptic cylinder, with the axis of symmetry parallel to $\mathbf{B}$. If $\mathbf{E} \cdot \mathbf{B} = 0$, this hodograph becomes a circle with radius $R = |\mathbf{E} + \mathbf{v}_0 \times \mathbf{B}|$, in a plane perpendicular to $\mathbf{B}$. Vector $\mathbf{v}$ sweeps the surface of a circular cone, which vertex falls in the circle plane if $\mathbf{v}_0 \cdot \mathbf{B} = 0$.

2. Formula (19) shows that $\mathbf{a}(0) = \mathbf{a}_0 = \frac{q}{m} (\mathbf{E} + \mathbf{v}_0 \times \mathbf{B})$; then $\mathbf{F}(0) = \mathbf{F}_0 = q(\mathbf{E} + \mathbf{v}_0 \times \mathbf{B})$, so $\mathbf{F}(t) = m\mathbf{a}$ is given by

$$(23) \quad \mathbf{F} = \frac{1}{B^2} (\mathbf{F}_0 \cdot \mathbf{B}) \mathbf{B} + \frac{1}{B} (\mathbf{F}_0 \times \mathbf{B}) \sin \omega t + \frac{1}{B^2} \mathbf{B} \times (\mathbf{F}_0 \times \mathbf{B}) \cos \omega t.$$

It reveals that the acting force, with constant modulus, having as hodograph a circle in a plane perpendicular to the magnetic lines, performs a uniform precession around these lines, with the angular speed $\omega^* = -\omega$. The force constant modulus is fixed at the

beginning of the interaction between the charged particle and the fields. The electric field prescribes the interaction intensity, while the magnetic field imposes its rhythm.

3. Motion of a Free Heavy Particle Relative to the Earth Surface.

Neglecting as usual the centrifugal force as compared to the Coriolis force, the equation of motion relative to the rotating earth ($\omega = \text{const.}$) reads [3], [5], [6]

$$(24) \quad \dot{\mathbf{v}} + 2\omega \times \mathbf{v} = \mathbf{g},$$

$\mathbf{v}$ and $\dot{\mathbf{v}}$ meaning now relative velocity and relative acceleration, respectively.

The solution of Eq. (24), identical with (17)₁, is (see (20))

$$(25) \quad \mathbf{v} = \frac{1}{2\omega^2} \mathbf{g} \times \omega + \frac{1}{\omega^2} (\mathbf{v}_0 \cdot \omega) \omega + \frac{1}{2\omega} (\mathbf{g} + 2\mathbf{v}_0 \times \omega) \sin 2\omega t + \\ + \frac{1}{2\omega^2} \omega \times (\mathbf{g} + 2\mathbf{v}_0 \times \omega) \cos 2\omega t + \frac{1}{2\omega^3} (\mathbf{g} \cdot \omega) (2\omega t - \sin 2\omega t) \omega.$$

The position vector results after integration and minor arrangements

$$(26) \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2\omega^2} (\omega \cdot \mathbf{g}) \omega t^2 + \frac{1}{4\omega^3} \omega \times (\mathbf{g} + 2\mathbf{v}_0 \times \omega) (\sin 2\omega t - 2\omega t) + \\ + \frac{1}{4\omega^4} \omega \times (\mathbf{g} \times \omega - 2\omega^2 \mathbf{v}_0) (1 - \cos 2\omega t).$$

For small ω , $\sin 2\omega t - 2\omega t \approx -\frac{4}{3}\omega^3 t^3 + \dots$, $1 - \cos 2\omega t \approx 2\omega^2 t^2$ and the terms of (26) containing the first three powers of t are

$$(27) \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \left(\frac{1}{2} \mathbf{g} + \mathbf{v}_0 \times \omega \right) t^2 + \frac{1}{3} (\mathbf{g} + 2\mathbf{v}_0 \times \omega) \times \omega t^3.$$

The penultimate term $\frac{1}{3} \mathbf{g} \times \omega t^3$ expresses the well known deviation from the vertical toward the east, currently obtained by a successive approximations technique [3], [6].

The formulae (25) or (26) are not presented in the literature.

4. The Reduced Problem of Foucault's Pendulum is described [3], [5], [6] in the uniformly rotating system by

$$(28) \quad \ddot{\mathbf{r}} + 2\omega \times \dot{\mathbf{r}} + \omega_0^2 \mathbf{r} = \mathbf{0},$$

where $\omega = \text{const.}$ is the earth angular velocity and $\omega_0^2 = g/l$, with g — gravitational acceleration and l — length of pendulum.

Let perform the vectorial product of (28) with ω , derivate the result, substitute the twice intervening there product $\omega \times \dot{\mathbf{r}}$ from the initial Eq. (28) and multiply finally again vectorially with ω to obtain

$$(29) \quad \frac{d^4}{dt^4} (\omega \times \mathbf{r}) + 2(\omega_0^2 + 2\omega^2) \frac{d^2}{dt^2} (\omega \times \mathbf{r}) + \omega_0^4 (\omega \times \mathbf{r}) = \mathbf{0}.$$

The roots of the characteristic equation are obviously imaginary

$$(30) \quad s_{1,2,3,4} = \pm i \left[\sqrt{\omega^2 + \omega_0^2} \pm \omega \right]$$

and the solution is

$$(31) \quad \omega \times \mathbf{r} = \mathbf{C}_1 \cos \omega_+ t + \mathbf{C}_2 \sin \omega_+ t + \mathbf{C}_3 \cos \omega_- t + \mathbf{C}_4 \sin \omega_- t,$$

where we put

$$(32) \quad \omega_+ = \omega^* + \omega, \quad \omega_- = \omega^* - \omega, \quad \omega^* = \sqrt{\omega^2 + \omega_0^2}.$$

The four constants are given by (31), its derivative, the Eq. (28) vectorially multiplied with ω and the derivative of this last result. With $\mathbf{u} = \omega/\omega$ they are

$$(33) \quad \begin{aligned} \mathbf{C}_1 &= \frac{1}{2\omega^*} \omega \times (\mathbf{u} \times \mathbf{v}_0 + \omega_- \mathbf{r}_0), & \mathbf{C}_2 &= \frac{1}{2\omega^*} \omega \times [\mathbf{v}_0 - \omega_- (\mathbf{u} \times \mathbf{r}_0)], \\ \mathbf{C}_3 &= \frac{1}{2\omega^*} \omega \times (-\mathbf{u} \times \mathbf{v}_0 + \omega_+ \mathbf{r}_0), & \mathbf{C}_4 &= \frac{1}{2\omega^*} \omega \times [\mathbf{v}_0 + \omega_+ (\mathbf{u} \times \mathbf{r}_0)]. \end{aligned}$$

Having in mind (31), Eq. (28) becomes

$$(34) \quad \ddot{\mathbf{r}} + \omega_0^2 \mathbf{r} = 2\omega_+ (\mathbf{C}_1 \sin \omega_+ t - \mathbf{C}_2 \cos \omega_+ t) + 2\omega_- (\mathbf{C}_3 \sin \omega_- t - \mathbf{C}_4 \cos \omega_- t),$$

with the obvious solution

$$(35) \quad \begin{aligned} \mathbf{r} &= \mathbf{K}_1 \cos \omega_0 t + \mathbf{K}_2 \sin \omega_0 t + \\ &+ \frac{1}{\omega} (-\mathbf{C}_1 \sin \omega_+ t + \mathbf{C}_2 \cos \omega_+ t + \mathbf{C}_3 \sin \omega_- t - \mathbf{C}_4 \cos \omega_- t). \end{aligned}$$

We get immediately

$$(36) \quad \mathbf{K}_1 = (\mathbf{u} \cdot \mathbf{r}_0) \mathbf{u}, \quad \mathbf{K}_2 = \frac{1}{\omega_0} (\mathbf{u} \cdot \mathbf{v}_0) \mathbf{u}$$

and with (32)_{1,2} the solution (35) goes to

$$\begin{aligned} \mathbf{r} &= \mathbf{K}_1 \cos \omega_0 t + \mathbf{K}_2 \sin \omega_0 t + \\ &+ \frac{1}{\omega} \left\{ [(\mathbf{C}_3 - \mathbf{C}_1) \sin \omega^* t + (\mathbf{C}_2 - \mathbf{C}_4) \cos \omega^* t] \cos \omega t - \right. \\ &\left. - [(\mathbf{C}_2 + \mathbf{C}_4) \sin \omega^* t + (\mathbf{C}_1 + \mathbf{C}_3) \cos \omega^* t] \sin \omega t \right\}. \end{aligned}$$

Now, having in mind (33) and (36)

$$(37) \quad \mathbf{r} = \left[\left[\mathbf{r}_0 \cos \omega_0 t + \frac{\mathbf{v}_0}{\omega_0} \sin \omega_0 t \right] \cdot \mathbf{u} \right] \mathbf{u} - \mathbf{u} \times (\mathbf{Y} \sin \omega t + \mathbf{Z} \cos \omega t),$$

where

$$\begin{aligned}
 (38) \quad Y &= r_0 \cos \omega^* t + \frac{1}{\omega^*} (v_0 + \omega \times r_0) \sin \omega^* t, \\
 Z &= (\mathbf{u} \times r_0) \cos \omega^* t + \frac{1}{\omega^*} (\mathbf{u} \times v_0 - \omega r_0) \sin \omega^* t.
 \end{aligned}$$

But it is easy to verify that $\mathbf{u} \times \mathbf{Z} = \mathbf{u} \times (\mathbf{u} \times \mathbf{Y})$, i.e. the solution to Eq. (28) is

$$(39) \quad \mathbf{r} = \left[\left[r_0 \cos \omega_0 t + \frac{v_0}{\omega} \sin \omega_0 t \right] \cdot \mathbf{u} \right] \mathbf{u} - \mathbf{u} \times (\mathbf{Y} \sin \omega t + \mathbf{u} \times \mathbf{Y} \cos \omega t).$$

The peculiarities of the Foucault pendulum behaviour are clearly expressed by the new obtained expression (39). The velocity hodograph can be obtained immediately.

Remarks. 1. An equation like (28) occurs in the *Classical Theory of Zeeman's Effect* [7].

2. Equation (28) describes also the motion of the vibrating element of a large class of gyroscopic devices [8].

3. It is obvious that the same method remains valid if in Eq. (28) the coefficient of the last term is negative.

5. The Complete Equation of a Heavy Particle Motion Relative to a Rotating Palnet is [3], [9]

$$(40) \quad \ddot{\mathbf{r}} + 2\omega \times \dot{\mathbf{r}} + \omega \times (\omega \times \mathbf{r}) = \mathbf{g},$$

which becomes homogeneous in the velocity ($\omega = \text{const.}$)

$$(41) \quad \ddot{\mathbf{v}} + 2\omega \times \dot{\mathbf{v}} + \omega \times (\omega \times \mathbf{v}) = \mathbf{0}.$$

Let derivate (41), replace $\ddot{\mathbf{v}}$ using (41) and multiply vectorial with ω to obtain

$$\omega \times \ddot{\mathbf{v}} + 3\omega^2 \omega \times \dot{\mathbf{v}} + 2\omega^2 \omega \times (\omega \times \mathbf{v}) = \mathbf{0}.$$

We derivate again and replace the vectorial product $2\omega \times \dot{\mathbf{v}}$ extracted from (41), to get

$$(42) \quad \frac{d^4}{dt^4} (\omega \times \mathbf{v}) + 2\omega^2 \frac{d^2}{dt^2} (\omega \times \mathbf{v}) + \omega^4 (\omega \times \mathbf{v}) = \mathbf{0},$$

which leads to

$$(43) \quad \omega \times \mathbf{v} = (C_1 + C_2 t) \cos \omega t + (C_3 + C_4 t) \sin \omega t.$$

The constants result

$$\begin{aligned}
 (44) \quad C_1 &= \omega \times v_0, & C_2 &= \omega \times (g + \omega^2 r_0 - \omega \times v_0), \\
 C_3 &= -\frac{1}{\omega} \omega \times (\omega \times v_0), & C_4 &= -\frac{1}{\omega} \omega \times \left[\omega \times (g + \omega^2 r_0) + \omega^2 v_0 \right].
 \end{aligned}$$

Returning to (41) we have $\ddot{\mathbf{v}} = \ddot{\mathbf{v}}(t)$ and a routine calculation gives ($\mathbf{u} = \omega/\omega$)

$$(45) \quad \mathbf{v} = (\mathbf{u} \cdot \mathbf{v}_0) \mathbf{u} - \mathbf{u} \times [\mathbf{v}_0 \sin \omega t - (\mathbf{u} \times \mathbf{v}_0) \cos \omega t] + (\mathbf{u} \cdot \mathbf{g}) \mathbf{u} t - \\ - (\mathbf{\Lambda} \sin \omega t + \mathbf{u} \times \mathbf{\Lambda} \cos \omega t) t, \quad \text{where} \quad \mathbf{\Lambda} = \mathbf{u} \times [\mathbf{g} - \omega \times (\mathbf{v}_0 + \omega \times \mathbf{r}_0)].$$

Finally

$$(46) \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2} (\mathbf{u} \cdot \mathbf{g}) \mathbf{u} t^2 - \\ - \frac{1}{\omega^2} \mathbf{u} \times [\mathbf{g} (\sin \omega t - \omega t \cos \omega t) - (\mathbf{u} \times \mathbf{g}) (1 - \cos \omega t - \omega t \sin \omega t)] - \\ - \mathbf{u} \times \left\{ [\mathbf{r}_0 \sin \omega t + (\mathbf{u} \times \mathbf{r}_0) (1 - \cos \omega t)] + [\mathbf{r}_0 \cos \omega t + (\mathbf{u} \times \mathbf{r}_0) \sin \omega t] \omega t \right\} + \\ + \frac{1}{\omega} \mathbf{u} \times [\mathbf{v}_0 \sin \omega t + (\mathbf{u} \times \mathbf{v}_0) (1 - \cos \omega t)] \omega t,$$

i.e. the motion law of the problem (40), given here for the first time.

6. Conclusion. 1. The vectorial differential equations (1)–(3) can be integrated under vectorial form, for various values of their coefficients.

2. Vectorial solutions are in this way established for a number of classical problems of theoretical mechanics, described by such equations and for which "exact solutions" are not reported.

3. Were considered here: the motion of charged particles in a magnetic field with fixed direction; the motion of charged particles in uniform magnetic and electric fields; the motion of a free heavy particle relative to the earth surface; the reduced problem of Foucault's pendulum; the complete equation of a heavy particle motion relative to a rotating planet.

3. A general and efficient symbolic method allowing to obtain with elegance the vectorial solutions of equations like (1) – (3), in conditions even less restrictive than those presented here, was worked out by one of the authors [10].

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STUDIUL DIRECT AL UNOR PROBLEME CLASICE DE MECANICĂ TEORETICĂ

(Rezumat)

Prin integrarea directă a ecuațiilor diferențiale vectoriale de tipul (1)–(3) se obțin soluții compacte, sub formă vectorială, nesemnificate în literatură, pentru o serie de probleme clasice de mecanică, dintre care se rețin aici: mișcarea sarcinilor electrice în câmpuri magnetice de direcție fixă; mișcarea sarcinilor electrice în câmpuri electric și magnetic uniforme; mișcarea relativă a particulei grele la suprafața Pământului; problema redusă a pendulului Foucault; problema generală a mișcării relative a particulei grele la suprafața unei planete în rotație.

A POSSIBLE GENERAL RELATIVISTIC EXPLANATION FOR THE GRAVITATIONAL ANOMALIES DURING THE SOLAR ECLIPSES

BY

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1. Introduction. Measurements made during the Solar eclipses showed some gravitational anomalies:

- i) a displacement of the oscillation plane of the *paraconic pendulum* [1], *Foucault pendulum* [4], [6], [8], *torsion pendulum* [10];
- ii) a shift of the newtonian gravitational field in the *Foucault pendulum*-type experiments [4], [6], [8].

Several assumptions have been made until now, such as: the Moon screening effect [8] or a modified newtonian force [10].

We try to explain these anomalies in a general relativistic model using the Higgs mechanism of a spontaneous symmetry breaking which modifies the gravitational index of refraction.

2. The Concrete Model. Let us assume that the Solar eclipses are deeply connected to a Higgs type cosmological field minimally coupled to space-time. We consider the following general metric for the spherical symmetric space-time [3], [5], [9]

$$(1) \quad ds^2 = n^{-1} c^2 dt^2 - n dr^2 - r^2 d\Omega.$$

According to our assumption, the spontaneous symmetry breaking induces a transition of the index of refraction from the empty space-time case [7]

$$(2) \quad n_0 \approx 1 + \frac{R_p}{r}, \quad R_p = \frac{2GM_p}{c^2}$$

to the case of a self-interacting scalar field as matter content [3], [5], [9]

$$(3) \quad n \approx 1 + \frac{R_p}{r} + \Lambda r^2, \quad \Lambda = \frac{\mu^4}{c^2},$$

where R_p is the Schwarzschild radius of the earth; μ and λ are the classical parameters of the ϕ^4 -model. Replacing (3) in (1) and using the relativistic Lagrangean for a test particle [7]

$$(4) \quad L = -m_0 c \left(\frac{ds}{dt} \right)$$

in the case of the newtonian approximation, one gets

$$(5) \quad L \approx \frac{1}{2} v^2 + \frac{1}{2} \frac{R_p c^2}{r} + \Lambda r^2 c^2.$$

The standard Euler-Lagrange procedure leads to a modified newtonian force

$$(6) \quad \mathbf{F} = -\frac{1}{2} \frac{R_p c^2}{r^2} \left(1 - \frac{2\Lambda}{R_p} r^3 \right) \frac{\mathbf{r}}{r}.$$

The non-newtonian gravitational term from (6) implies the following relativistic shift of the newtonian gravitational field:

$$(7) \quad \frac{\Delta g}{g} = -\frac{2\Lambda}{R_p^2} R^3,$$

where R is the earth radius.

The following remark might be of some interest. If we consider

$$(8) \quad \Lambda = -\frac{1}{R_0^2},$$

where R_0 is a characteristic radius for the Solar system, i.e. 10^{13} m, and

$$(9) \quad R_p \approx 10^{-2} \text{ m}, \quad R \approx 6.4 \cdot 10^6 \text{ m},$$

the newtonian gravitational shift gets the magnitude

$$(10) \quad \frac{\Delta g}{g} = \frac{2R^3}{R_0^2 R_p} \approx 10^{-4},$$

which is in a perfect agreement with the experimental data [4], [6], [10]. It could be also noticed that the well-known expression

$$(11) \quad \tau = nc^{-1}d,$$

d being the Sun — Earth distance, is naturally contained in our proposed model. It has the concrete value $\tau = 8$ min which could be interpreted as the observed delay in the shift of the newtonian gravitational field with respect to the beginning of the eclipse.

3. Conclusions. The Λ parameter, equal to $\mu^4 \lambda$, gets the meaning of a coupling constant between the Higgs and the gravitational fields. Thus, a correspondence with the Mach's principle is achieved, at least locally.

For

$$(12) \quad \Lambda = -\frac{\omega^2}{c^2},$$

one can think of the Higgs type cosmological field being the additional scalar field introduced in the Gödel model in order to eliminate the non-inertial effects.

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O EXPLICAȚIE POSIBILĂ PRIN RELATIVITATEA GENERALĂ A ANOMALIILOR GRAVITAȚIONALE OBSERVATE ÎN TIMPUL ECLIPSELOR DE SOARE

(Rezumat)

Utilizând mecanismul Higgs de rupere spontană a simetriei câmpului gravitațional de metrică Schwarzschild, se explică anomaliile gravitaționale observate în timpul eclipselor totale de Soare: creșterea aparentă a accelerației gravitaționale a unui pendul Foucault, devierea planului de oscilație a aceluiași pendul etc.

$$A = \frac{W}{C}$$

(11)

one can think of the integrals components field being the additional scalar field introduced in the Eddington model in order to eliminate the non-physical effects

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O EXPLICAȚIE POUȚĂ A PRIN BILANȚUL GENERAL A ANOMALIILOR GRAVITAȚIONALE CURENȚE ÎN TIMPUL SOLARILOR DE SOARE

(Romanian)

Un model relativist al câmpului gravitațional este prezentat în acest articol. Modelul este construit pe baza principiilor de relativitate și de conservare a energiei și impulsului. Se arată că modelul este în acord cu observațiile experimentale și cu rezultatele teoretice obținute în cadrul teoriei relativității generale.

THE SPIN OF THE KERR FIELD

BY

SERVILIA OANCEA

1. The introduction of the spin in the theories of gravitation is necessary in order to extend the general relativity to microphysics, where the spin is a fundamental quantity [1], [2]. The gauge theories of gravitation point out that the degrees of freedom are related to the spin of the particles [3].

In the present paper one gives some results about the spin of the Kerr field in the context of U_4 -theory. The Newman-Penrose formalism is used to establish a physical significance of the field spin angular momentum.

2. The microscopic field equations of U_4 -theory are

$$(1) \quad R_{ij} - \frac{1}{2} g_{ij} R = k \Sigma_{ij},$$

$$(2) \quad S_{ij}{}^k + \delta_i{}^k S_{jl}{}^l - \delta_j{}^k S_{il}{}^l = k \tau_{ij}{}^k,$$

where k is the relativistic gravitational constant, $\delta_i{}^j$ — the Kronecker delta; R_{ij} — the Ricci tensor; Σ_{ij} and $\tau_{ij}{}^k$ are the canonical energy-momentum and spin angular momentum of the matter field, respectively.

Because $\tau_{ij}{}^k$ is the density of the spin angular momentum, the spin angular momentum is obtained by integration, i.e.

$$(3) \quad S_{ij} = \int \tau_{ij}{}^k ds_k.$$

If we suppose fixed t , then the spin is given by the relation

$$(4) \quad S_{\alpha\beta} = \int \tau_{\alpha\beta}{}^4 ds_4, \quad \alpha, \beta = 1, 2, 3; \quad ds_4 = dx^1 dx^2 dx^3.$$

In (1) $S_{ij}{}^k$ is the torsion tensor

$$(5) \quad S_{ij}{}^k = \frac{1}{2} (\Gamma_{ij}{}^k - \Gamma_{ji}{}^k)$$

and $\Gamma_{ij}{}^k$ is the Riemann-Cartan connection.

For a differentiable manifold with absolute parallelism, the connection is

$$(6) \quad \Gamma_{ij}{}^k = e_a{}^k e_{j,i}{}^a,$$

where $e_a{}^k$ and $e_j{}^a$ are the null tetrads.

Let be a Kerr space-time. For this space we propose the followings null-tetrads:

$$(7) \quad \begin{aligned} e_1^j &= \frac{1-i}{2} \left\{ \delta_1^j + i \left[\delta_4^j - \left(\frac{1}{2} - \frac{mr - e^2/2}{r^2 + a^2 \cos^2 \theta} \right) \delta_1^j \right] \right\}, \\ e_2^j &= \frac{1-i}{2} \left[i \delta_1^j + \delta_4^j - \left(\frac{1}{2} - \frac{mr - e^2/2}{r^2 + a^2 \cos^2 \theta} \right) \delta_1^j \right], \\ e_3^j &= \frac{i\sqrt{2}}{2} \left\{ \frac{1}{r^2 + a^2 \cos^2 \theta} \left[a \sin \theta (r + a \cos \theta) (\delta_4^j - \delta_1^j) + \right. \right. \\ &\quad \left. \left. + \delta_2^j (r - a \cos \theta) + \frac{\delta_3^j}{\sin \theta} (r + a \cos \theta) \right] \right\}, \\ e_4^j &= -i\sqrt{2} \left\{ \frac{1}{r^2 + a^2 \cos^2 \theta} \left[a \sin \theta (r - a \cos \theta) (\delta_4^j - \delta_1^j) - \right. \right. \\ &\quad \left. \left. - \delta_2^j (r + a \cos \theta) + \frac{\delta_3^j}{\sin \theta} (r - a \cos \theta) \right] \right\}. \end{aligned}$$

The non-null components of the spin angular momentum tensor can be calculated by (6), (5) and (2). They are

$$(8) \quad \begin{aligned} \tau_{12}^1 &= \frac{1}{2k} \cotan \theta \left[1 - \frac{a^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \right], \quad \tau_{13}^1 = \frac{1}{2k} \frac{a r \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} = \tau_{13}^4, \\ \tau_{23}^1 &= \frac{1}{2k} \frac{a \sin \theta \cos \theta}{(r^2 + a^2 \cos^2 \theta)^2} \left[(r^2 + a^2)(r^2 + a^2 \cos^2 \theta) + \sin^2 \theta (2mr - e^2) \right], \\ \tau_{24}^1 &= -\frac{1}{2k} \frac{a^2 \sin \theta \cos \theta (2mr - e^2)}{r^2 + a^2 \cos^2 \theta}, \quad \tau_{12}^2 = \tau_{13}^3 = -\frac{1}{2k} \frac{r}{r^2 + a^2 \cos^2 \theta}, \\ \tau_{13}^2 &= -\frac{1}{2k} \frac{a \sin \theta \cos \theta}{r^2 + a^2 \cos^2 \theta}, \quad \tau_{12}^3 = \frac{1}{2k} \frac{a \cotan \theta}{r^2 + a^2 \cos^2 \theta}, \\ \tau_{23}^2 &= \frac{1}{2k} a \sin^2 \theta \left[\frac{m(r^2 - a^2 \cos^2 \theta) - r e^2}{(r^2 + a^2 \cos^2 \theta)^2} + \frac{r}{r^2 + a^2 \cos^2 \theta} \right], \end{aligned}$$

$$\tau_{24}^2 = -\frac{1}{2k} \frac{m(r^2 - a^2 \cos^2 \theta) - re^2}{(r^2 + a^2 \cos^2 \theta)^2}, \quad \tau_{24}^4 = -\frac{1}{2k} \cotan \theta,$$

$$\tau_{23}^3 = -\frac{1}{2k} \frac{a^2 \sin \theta \cos \theta}{r^2 + a^2 \cos^2 \theta} = -\tau_{12}^4, \quad \tau_{14}^4 = -\frac{1}{k} \frac{r}{r^2 + a^2 \cos^2 \theta},$$

$$\tau_{23}^4 = -\frac{1}{2k} a \sin \theta \cos \theta \frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta},$$

$$\tau_{34}^3 = -\frac{1}{2k} \frac{m(r^2 - a^2 \cos^2 \theta) - re^2}{(r^2 + a^2 \cos^2 \theta)^2},$$

$$\tau_{34}^4 = -\frac{1}{2k} a \sin^2 \theta \frac{m(r^2 - a^2 \cos^2 \theta) - re^2}{(r^2 + a^2 \cos^2 \theta)^2}.$$

Introducing (8) in (4) we obtain that $S_x = S_{23} = 0$, $S_y = S_{13} = 0$, $S_z = S_{12} = 0$. Using tensor-integral we can verify that the spin of the field is null.

The Newman-Penrose formalism [4] is used to study the space-time with absolute parallelism and to obtain the relation between the density of spin angular momentum and the physical properties of the body. Using the passing of the coordinate basis to a tetradic one, we obtain the tetradic components of the spin angular momentum.

The non-null components are:

$$k\tau_{12}^1 = -\frac{r}{r^2 + a^2 \cos^2 \theta}, \quad k\tau_{23}^1 = \frac{1}{2\sqrt{2}} \frac{\cotan \theta}{r + ia \cos \theta} = k\bar{\tau}_{24}^1 = -k\tau_{14}^2,$$

$$k\tau_{12}^2 = -\frac{r}{r^2 + a^2 \cos^2 \theta} \left[\frac{1}{2} - \frac{mr - e^2/2}{r^2 + a^2 \cos^2 \theta} \right],$$

$$(9) \quad k\tau_{34}^2 = \frac{ia \cos \theta}{4(r^2 + a^2 \cos^2 \theta)} \left[1 + \frac{2mr - e^2}{r^2 + a^2 \cos^2 \theta} \right],$$

$$k\tau_{14}^3 = \frac{1}{2(r + ia \cos \theta)} = k\bar{\tau}_{13}^4, \quad k\tau_{34}^1 = \frac{ia \cos \theta}{r^2 + a^2 \cos^2 \theta},$$

$$k\tau_{23}^2 = \frac{1}{2\sqrt{2}} \frac{a \sin \theta \left\{ [im(r^2 - a^2 \cos^2 \theta) - re^2] + a \cos \theta (2mr - e^2) \right\}}{(r + ia \cos \theta)(r^2 + a^2 \cos^2 \theta)^2} = k\bar{\tau}_{24}^2,$$

$$k\tau_{23}^4 = -\frac{1}{2(r-i a \cos \theta)} \left[\frac{1}{2} - \frac{mr - e^2/2}{r^2 + a^2 \cos^2 \theta} \right] - \frac{m(r^2 - a^2 \cos^2 \theta) - re^2}{(r^2 + a^2 \cos^2 \theta)^2} = k\tau_{24}^3.$$

3. In [5] we expressed the components of the torsion tensor as function of the Newman-Penrose *spin coefficients*. Taking into account the relation between torsion and spin, we get for the spin the following expressions:

$$\begin{aligned} k\tau_{34}^1 &= \frac{1}{2}(\bar{\rho} - \rho), & k\tau_{12}^1 &= \frac{1}{2}(\rho + \bar{\rho}), & k\tau_{12}^2 &= \frac{1}{2}(\bar{\mu} + \mu), & k\tau_{23}^2 &= \nu, \\ k\tau_{23}^1 &= \frac{1}{2}(\beta - \bar{\alpha}), & k\tau_{24}^3 &= \frac{1}{2}(\mu - 2\gamma), \\ (10) \quad k\tau_{14}^2 &= \frac{1}{2}(\bar{\beta} - \alpha), & k\tau_{34}^2 &= \frac{1}{2}(\mu - \bar{\mu}), \\ k\tau_{14}^3 &= \frac{1}{2}(\bar{\rho} + \varepsilon - \bar{\varepsilon}). \end{aligned}$$

Thus, all the components of the spin are expressed as combinations of the Newman-Penrose "spin coefficients". This fact may justify the name of these coefficients. We can say that the Newman-Penrose "spin coefficients" are related to the spin of the particles that produce the gravitational field.

The component τ_{34}^1 is

$$(11) \quad 2k\tau_{34}^1 = \frac{2ia \cos \theta}{r^2 + a^2 \cos^2 \theta} = -\rho + \bar{\rho}.$$

But $\rho = -\theta + i\omega$, where θ is the expansion and ω is the twist. Then τ_{34}^1 becomes

$$(12) \quad 2k\tau_{34}^1 = -2i\omega, \quad \omega = -\frac{a \cos \theta}{r^2 + a^2 \cos^2 \theta}.$$

Thus τ_{34}^1 , like S_{34}^1 component, is related to the twist motion of the body that produces the gravitational field.

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SPINUL CÂMPULUI KERR

(Rezumat)

Se calculează componentele tensorului momentului cinetic de spin pentru câmpul Kerr și se arată că una dintre aceste componente este legată de mișcarea de "twist" a corpului ce produce câmpul gravitațional.

A NEW METHOD FOR CHARGE STABILITY STUDY IN DIELECTRICS

BY

EUGEN NEAGU and RODICA NEAGU

1. Introduction. The study of charge stability in dielectrics is of great interest for fundamental point of view as well as for applications. The method of Thermally Stimulated Discharge Current is used for the study of charge traps in dielectrics [1], [2]. If the charge in the dielectric is determined by dipole orientation, the released current in external circuit on the two sides of the sample are equals and have no importance on which face of the sample the current is measured. Because always the dipolar charge is accompanied by charges proceeding from other mechanisms, especially in the case of α and ρ peaks, we consider that the two currents are not equal, and it is necessary to measure the currents released by the both sides of the sample.

The aim of this paper is to present our results concerning the thermally stimulated discharge currents measured on both sides of a sample of high density polyethylene containing an antioxidant. The stability of positive charge is analyzed.

2. Experimental Procedure. If polyethylene contains different antioxidants it is more stable and no oxidation products appear in the sample, that is no polar groups are formed by oxidation. The polymer is partially crystalline and the intrinsic excess charge will mainly pile up at the phase boundaries.

The sample was $7.5 \cdot 10^{-5}$ m thickness with 0.964 g/cm^3 density. They were cleaned and aluminium electrodes of $5 \cdot 10^{-2}$ m diameter were vacuum evaporated on their surfaces. The sample was mounted in the sample chamber which was subsequently evacuated to a pressure of 10^{-4} torr (Fig. 1). The sample was heated to T_p under a bias voltage V_p and the field was applied for a period time t_p . After this time the sample is cooled at room temperature and the bias voltage removed. The sample is maintained in short-circuit for a time, after which the temperature is slowly increased at a constant heating rate of 1.5°C/min , to a temperature near the melting point of the sample.

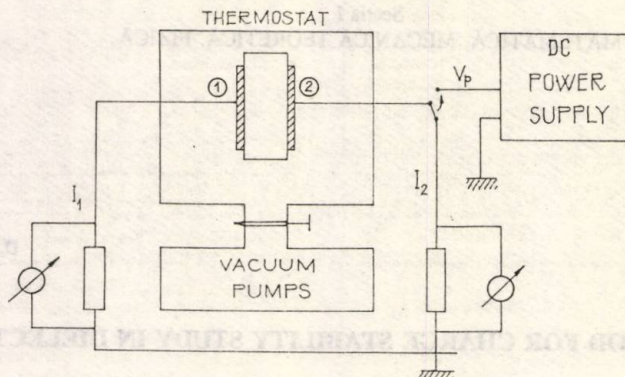


Fig. 1

3. Results and Discussion. Fig. 2 shows some typical thermograms for polyethylene containing an antioxidant. All TSDC measurements were made on the same sample. Before these measurements, on the same sample have been made TSDC measurements with the polarization field negative, that is the face two of the dielectric had been connected with the negative end of d.c. supply. Before the new experiment the sample have been heated three times to 110°C to remove the less persistent charge. Curve 1 in Fig. 2 represents the thermogram for the case when the sample was polarized at 85 kV/cm for 1 h at 90°C . The released current are of opposite sign even if the sample is not polar. When the temperature is bigger than 80°C the released current from the face two starts to decrease, charge sign and a negative maximum was obtained at 114°C . In the same time the current released from face one is negative and have the main maximum at the same temperature.

We have observed [3], [4] that the negative charge have a good stability. To prove this conclusion we have maintained the sample 9 days at room temperature in the sample holder. After this time a new TSDC experiment was performed. The results are presented by curves 2 in Fig. 2. The currents change the sign at about 60°C . From the figure we can see that the current released from the face one is still small but the current released from the face two increases very much at about the same temperature as in the case of negative polarization field [5]. From these results we can conclude that the negative charge has a better stability than the positive charge. In the same time we can say that it is necessary to make a very good "electrical cleaning" of the sample before a TSDC experiment.

In order to obtain more informations about positive charge stability we have increased the polarization temperature to 100°C . The results are presented by curves 3 in Fig. 2. It is seen that the initial currents are smaller than that corresponding in curves 1. This is because we have studied initially the isothermal charge decay for 15

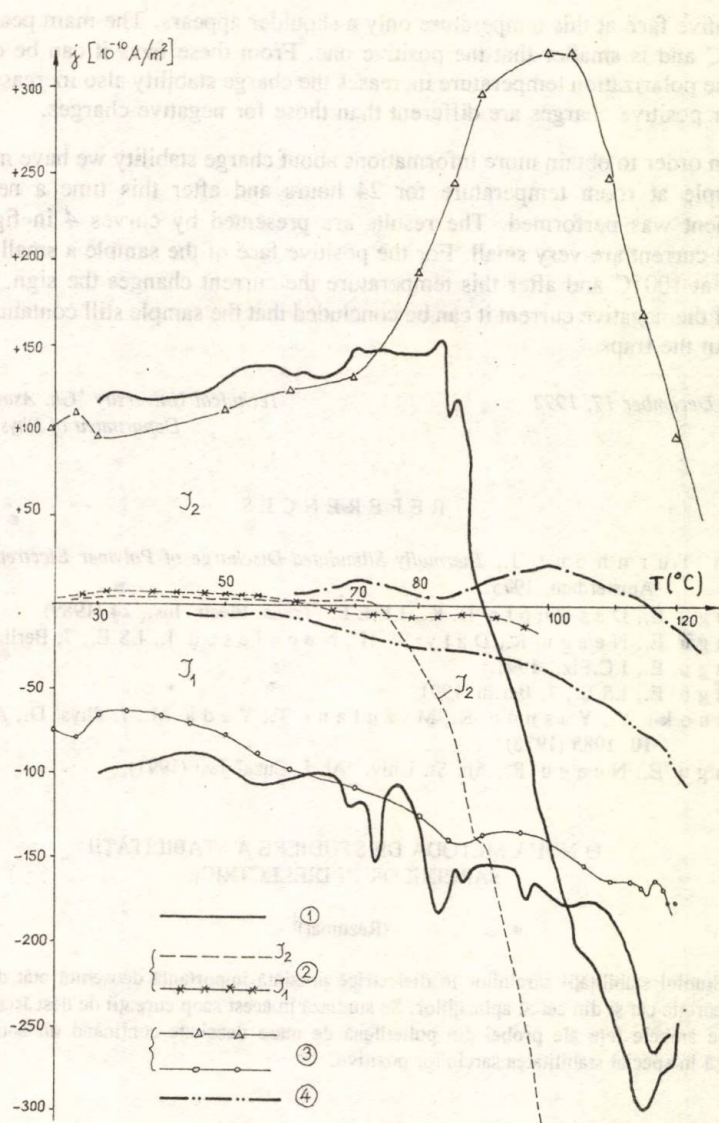


Fig. 2

minutes in the first case and for 40 minutes in this case. The first peak appears at 27°C for both faces and is specific for polyethylene [6], [7]. The main peak for the positive face appears at polarization temperature and therefore it is a ρ type peak. For

the negative face at this temperature only a shoulder appears. The main peak appears at 112°C and is smaller than the positive one. From these facts it can be concluded that if the polarization temperature increases the charge stability also increases and the traps for positive charges are different than those for negative charges.

In order to obtain more informations about charge stability we have maintained the sample at room temperature for 24 hours and after this time a new TSDC experiment was performed. The results are presented by curves 4 in fig. 2. The released current are very small. For the positive face of the sample a small peak still appears at 100°C and after this temperature the current changes the sign. From the shape of the negative current it can be concluded that the sample still contains negative charge in the traps.

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O NOUĂ METODĂ DE STUDIERE A STABILITĂȚII SARCINILOR ÎN DIELECTRICE

(Rezumat)

Studiul stabilității sarcinilor în dielectrice prezintă importanță deosebită atât din punct de vedere teoretic cât și din cel al aplicațiilor. Se studiază în acest scop curenții de descărcare stimulați termic pe ambele fețe ale probei din polietilenă de mare densitate conținând un antioxidant. Se analizează în special stabilitatea sarcinilor pozitive.

PHOTOANODES OF $\alpha\text{-Fe}_2\text{O}_3$ IMPURIFIED WITH NIOBIUM

BY

SERGIU ISTRATE and EUGEN NEAGU

1. Introduction. Photoelectrolysis of water could provide conversion of the solar energy to chemical energy in the photoelectrolytic process, and storage of this energy as gaseous or liquid oxygen and hydrogen. Photoelectrolysis occurs in cells with a light-absorbing n -type semiconductor as anode and a platinum cathode. The electrolyte used in these cells has usually been an aqueous solution with an ionic strength sufficient to conduct the current with small losses.

The anode is usually a metal oxide and three primary requirements are needed for such an electrode: a) its band gap should be of the order of 2.0 eV so as to utilize a large portion of the solar spectrum; b) the cell should operate effectively at zero bias so as to obviate the need for an external power supply; c) the semiconductor must be inert under any conceivable chemical or photoelectrochemical conditions encountered in the cell.

In this paper we present a potential photoelectrode $\alpha\text{-Fe}_2\text{O}_3$. Wickersheim and Leffever [1] studying the absorption spectra of the simple ferric oxide, conclude that in Fe_2O_3 appear absorptions at $12\,000\text{ cm}^{-1}$ and $16\,000\text{ cm}^{-1}$, which correspond to the activation energies of processes 1.48 and 1.99 eV respectively.

We realized samples of $\alpha\text{-Fe}_2\text{O}_3$ impurified with niobium, since the pure ferric oxide has a very small electrical conductivity [2].

2. Experimental. 2.1. Method of Preparation. The Fe_2O_3 (Riedel de Haen AG Seelze, Hanover—pure) powder was mixed with Nb_2O_5 (Fluka AG — puriss.) powder in a wet ball mill for two hours. The slurry was at once dried in a drying closet at 90°C . The mixed powder was dry pressed into pellets of typical dimensions 10 mm (diameter) $\times 3\text{ mm}$ (thick).

The pellets were sintered in air at 1200°C , during 24 hours (treatment T_5) and 9 hours (treatment T_8) respectively, and then slowly cooled in the furnace. After the treatment performed in air we observed that the samples present a superficial layer with great electrical resistance, in accord with paper [3]. Usually this layer was

eliminated by grinding.

The structure of our probes was determined with the aid of DRON-2,0 diffractometer, utilising $\text{FeK}_{\alpha\beta}$ radiation. The obtained structure results, according to "Inorganic Index to the Powder Diffraction File" 1965 and card index ASTM respectively, as being $R\bar{3}c$. It was not observed other phases in the sintered materials.

The impurity levels was empirically determined. In the Table 1 we present the symbols for our probes.

Table 1
The probes symbols

Level impurity Nb_2O_5 %	Treatment	Symbol	Treatment	Symbol
0.7	T_5	14	T_8	N_1
0.4		15		N_2
0.1		16		N_3
0.07		17		N_4

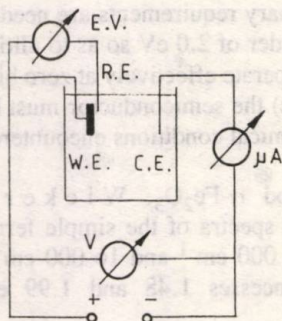


Fig. 1 — Photoelectrochemical cell: E.V. — electronic voltmeter; R.E. — reference electrode; W.E. — work electrode; C.E. — counter electrode; V — voltmeter; μA — microammeter.

2.2. Measurements. Current-potential curves in the dark and under illumination for our semiconductor electrodes were determined with the aid of a photoelectrochemical cell. The reference electrode was a saturated Calomel electrode (SCE) and the counter electrode a platinum foil. The work electrode was a pellet of $\alpha\text{-Fe}_2\text{O}_3$ impurified. Electrical contacts to the $\alpha\text{-Fe}_2\text{O}_3$ impurified was made on the face polished using very fine emery paper, with In-Ga paste. The Fe_2O_3 pellet was sealed to a black support by epoxy resin. The other face fine polished too, was in contact with the electrolyte. The window of the cell was quartz glass and illumination accomplished with a 200 W mercury lamp. The electrochemical photo cell is presented schematically in Fig. 1. The electronic voltmeter was of type E-0302 (IEMI-Bucharest) and the electrolyte 1 M NaOH. The external applied voltage was comprised between 0...1.4 V.

As flat-band potential U_{fb} we understand the value of the semiconductor electrode potential relating to reference electrode, when the band bending vanishes and $V_B=0$, where the voltage V_B is the band-bending or the potential drop across the space charged region. V_B can be varied when one applies an external voltage between work electrode and counter electrode

respectively. Since V_B is the Galvani potential difference between the surface and bulk of the semiconductor, we cannot directly measure U_{fb} ; it can be estimate with the method presented by B a b a i and coworkers [4], but this method is not accurate when the electrodes are polycrystalline [5].

A second method is based on the photoresponse of semiconductor-electrolyte junction [6] when it is irradiated with light of appropriate wavelength. The I_{ph}^2 (I_{ph} -photocurrent) versus V characteristics exhibits a linear part whose extrapolation to $I_{ph}^2 = 0$ gives the U_{fb} value.

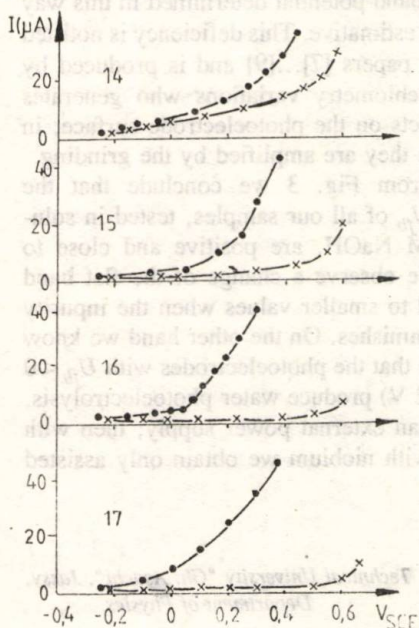


Fig. 2 — The current-potential curves for α - Fe_2O_3 impurified with niobium; treatment T_5 .

then the electrical circuit is practically interrupted. This fact was observed by us in a preliminary stage when we realized the impurified samples with 1 ; 0.1 and 0.01 % respectively; the samples doped with 0.01 % have not presented photoresponse.

Therefore we see that the photoelectrodes claim two antagonistic requirements: a strong impurification for electrical resistance to be as small as possible and a weak one for to obtain a sufficiently large depletion region. A possible solution for the realization of the corresponding photoelectrodes is a preparation method for samples with reduced structural defects and as far as possible the grain boundary not to be oxidized.

3. Results and Discussion. The current — potential curves were determined with the aid of the installation presented in Fig. 1. In the Fig. 2 we represent these characteristics for samples obtained by T_5 treatment, where we utilize the symbol ● for the photocurrent, and × respectively, for the dark current.

It can be seen from the Fig. 2 that in the samples with small impurity levels — 0.1 and 0.07 at %, respectively — the effect of semiconductor irradiation is strong, resulting the efficient separation of the carriers, and then a notable photocurrent, while the dark current remains practically zero. In the samples doped with 0.4 and 0.7 % respectively, the carrier concentration is larger, the depletion region is relatively narrow and then the efficient carrier separation is too small. The photocurrent varies alike the dark current as a consequence of the external applied voltage, when it modifies. On the other hand an impurification level too small it not conveniently, because the electrical resistance of the electrode is very large and

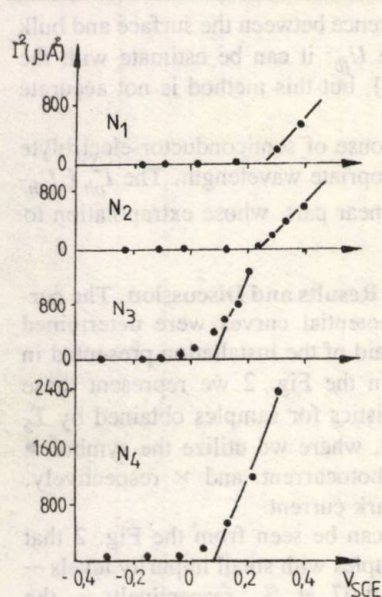


Fig. 3 — The detrmintion of U_{fb} ; treatment T_8 .

In Fig. 3 we represent $I_{ph}^2 = f(V_{SCE})$ for samples obtained by T_8 treatment. Because $|V_{on}| < |U_{fb}|$, where V_{on} is the potential corresponding to the onset of the photocurrent, we extrapolated the portion of the curve in which I_{ph}^2 varies too strongly. Since the increase of the photocurrent is not abrupt from zero value, it results that the values of the flat band potential determined in this way are only estimative. This deficiency is notified in other papers [7]...[9] and is produced by the stoichiometry variations who generates the defects on the photoelectrode surface; in our case they are amplified by the grinding.

From Fig. 3 we conclude that the values U_{fb} of all our samples, tested in solution 1 M NaOH, are positive and close to zero. We observe a change of the flat band potential to smaller values when the impurity level diminishes. On the other hand we know [5], [10] that the photoelectrodes with $U_{fb} < 0$ (≈ -1.2 V) produce water photoelectrolysis, without an external power supply; then with

the photoelectrodes from $\alpha\text{-Fe}_3\text{O}_2$ impurified with niobium we obtain only assisted photoelectrolysis.

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FOTOANOD DE $\alpha\text{-Fe}_2\text{O}_3$ IMPURIFICAT CU NIOBIU

(Rezumat)

Având în vedere că fotoelectroliza apei poate asigura conversia energiei solare în energie chimică care poate fi stocată, fotoelectroliză care utilizează în mod curent anodi din oxizi metalici, se realizează și se studiază performanțele unui fotoanod din $\alpha\text{-Fe}_2\text{O}_3$ impurificat cu niobiu.

PHOTOGRAPH OF A Fe_2O_3 IMPURIFIED Cu NIOBATE

(Figure 1)

Figure 1 shows a photograph of a Fe_2O_3 impurified Cu NIOBATE. The sample is a dark, irregularly shaped solid. The photograph is taken against a light background. The sample appears to be a single piece of material, possibly a crystal or a sintered pellet. The surface of the sample is somewhat rough and uneven. The color is a dark, almost black, with some lighter, brownish areas visible, particularly towards the edges. The overall shape is roughly rectangular but with irregular corners and edges. The photograph is oriented horizontally on the page.

PERMEABILITY SPECTRUM IN FERRITES

BY

DORIN CONDURACHE and I. GROSU

1. The frequency spectrum of the complex initial permeability offers important informations regarding the field of applications of the corresponding magnetic material. The resonance frequency (the permeability dispersion frequency) indicates the upper limit of utilization.

In this paper we propose an "elastic membrane" like model of Bloch walls in order to compute the complex magnetic permeability.

2. It is well known that a coil with magnetic core is characterized by the electric impedance

$$(1) \quad Z = R + i\omega L,$$

while for the same coil without core

$$(2) \quad Z_0 = R_0 + i\omega L_0;$$

the frequency $\nu = \omega/2\pi$ of the magnetic field produced by the winding is the same in both cases.

The core quality factor is given by

$$(3) \quad Q = \frac{L'\omega}{R'},$$

where $L' = L - L_0$ and $R' = R - R_0$; it is connected with the total core losses by means of the expression

$$(4) \quad Q = \frac{1}{\tan \delta},$$

where δ is the loss angle.

A good quality factor implies a high apparent permeability of the core $\mu_a = L/L_0$ and a small loss resistance.

3. For a polycrystalline magnetic material with relatively large grains (polydromenic) with 180° Bloch walls the macroscopic magnetization vanishes for a zero external field. After applying the magnetizing field H , the magnetization increases

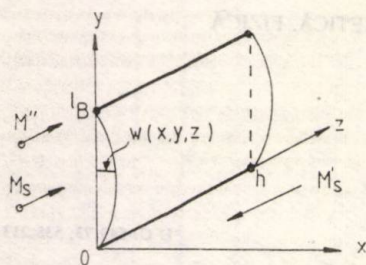


Fig. 1 — A momentary position of the Bloch wall.

given the partial spin orientation and the Bloch wall movement, the last process bringing a contribution of about 80...85% to the total magnetic permeability. At higher frequency the Bloch walls can no longer move but they bend like elastic membranes with fixed edges and perform a forced oscillatory movement with the frequency equal to that of the magnetizing field.

The momentary position of a wall with the length l_B and the width h is given in Fig. 1.

The magnetization of the microscopic sample, calculated under the above condition, is given by the expression [1]

$$(5) \quad M = 8\mu_0 M_s^2 \frac{N h l_B}{\pi^2 m} H \sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)^2}{n^2 \omega_n^2 (1 - \omega^2/\omega_n^2 + i2\varepsilon\omega/\omega_n^2)},$$

where M_s is the intrinsic saturation magnetization; N — the number of Bloch walls in an unit volume; m — the mass of the wall unit surface; ε — the damping coefficient of the wall oscillation whose resonance frequency is ω_n .

The relation (5) is a typical dispersion relation. Since

$$(6) \quad M = \chi H = (\mu' - 1 + i\mu'') H,$$

one can find once μ' , the real, and μ'' , the imaginary components of the complex magnetic permeability. These are:

$$(7) \quad \mu' = 1 + 8\mu_0 M_s^2 \frac{N h l_B^3}{\pi^4 \gamma} \sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)^2 (1 - \Omega^2)}{n^4 [(1 - \Omega^2)^2 + A_n^2 \Omega^2]}$$

and

$$(8) \quad \mu'' = 8\mu_0 M_s^2 \frac{N h l_B^3}{\pi^4 \gamma} \sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)^2 A_n \Omega}{n^4 [(1 - \Omega^2)^2 + A_n^2 \Omega^2]},$$

where γ is the elastic coefficient of the "Bloch membrane"; $\Omega = \omega/\omega_n$ and $A_n = 2\varepsilon/\omega_n$. The series we obtain are quickly converging so that the first two terms give a quite good approximation.

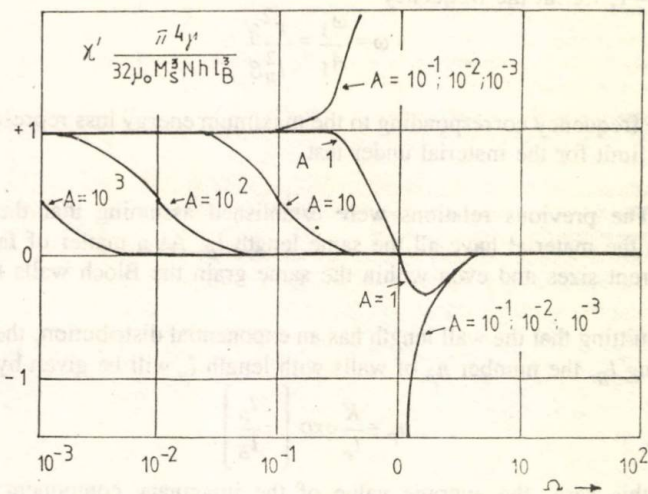
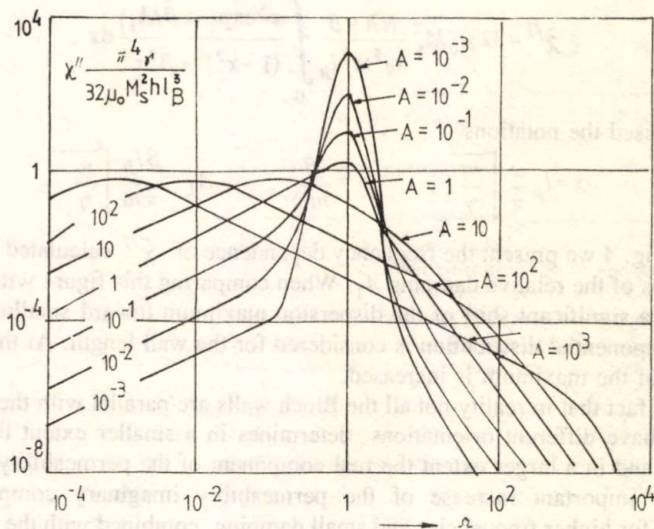
In Figs. 2 and 3 the frequency dependences of these terms are given for different values of the relative damping A .

In the case of small damping ($A_1 < 1$) the maximum energy loss will be obtained at the frequency

$$(9) \quad \omega = \omega_1 = \frac{\pi}{l_B} \sqrt{\frac{\gamma}{m}},$$

while for large damping $A_1 \gg 1$ and $\Omega \ll 1$, the relation (8) can be written as

$$(10) \quad \mu' = 32\mu_0 M_s^2 \frac{N h l_B^2}{\pi^4 \gamma} \frac{A_1 \Omega}{1 + A_1^2 \Omega^2}$$

Fig. 2 — χ' vs. frequency for different damping A .Fig. 3 — χ'' vs. frequency for different damping A .

The imaginary component μ'' of the complex permeability has the maximum

$$(11) \quad \mu'' = 16 \mu_0 M_s^2 \frac{N h l_B^2}{\pi^4 \gamma},$$

when $A_1 \Omega = 1$, i.e. at the frequency

$$(12) \quad \omega = \frac{\omega_1}{A_1} = \frac{\pi^2 \gamma}{l_B^2 \beta}.$$

The frequency corresponding to the maximum energy loss represents the upper utilization limit for the material under test.

4. The previous relations were established assuming that the Bloch walls existing in the material have all the same length l_B . As a matter of fact, the grains have different sizes and even within the same grain the Bloch walls have different lengths.

Admitting that the wall length has an exponential distribution, the average wall length being l_B , the number n_p of walls with length l_p will be given by

$$(13) \quad n_p = \frac{K}{l_p} \exp \left[-\frac{l_p}{l_B} \right].$$

In this case, the average value of the imaginary component of the initial complex magnetic susceptibility will be given by

$$(14) \quad \bar{\chi}'' = 32 \mu_0 M_s^2 \frac{N h \gamma \beta}{\omega^5 m^3 l_B} \int_0^{\infty} \frac{x^5 \exp(-x \bar{\beta} / A_1)}{(1-x^2)^2 + \bar{\beta}^2 x^4} dx,$$

where we used the notations

$$x = l_p \frac{\omega}{\pi} \sqrt{\frac{m}{\gamma}}, \quad \bar{\beta} = \frac{\beta}{m \omega}, \quad A_1 = \frac{\beta l_B}{\pi m} \sqrt{\frac{m}{\gamma}}.$$

In Fig. 4 we present the frequency dependence of $\bar{\chi}''$ calculated with (14) for three values of the relative damping A_1 . When comparing this figure with Fig. 3, one can notice a significant shift of the dispersion maximum toward smaller frequencies when an exponential distribution is considered for the wall length. At the same time, the width of the maximum is increased.

The fact that in reality not all the Bloch walls are parallel with the magnetizing field, but have different orientations, determines in a smaller extent the dispersion frequency and in a larger extent the real component of the permeability.

The important increase of the permeability imaginary component, more significant for higher frequencies and small damping, combined with the simultaneous decrease of its real component, determines a certain working frequency of the magnetic core, the weaker is the wall bonding (the smaller in the magnetic anisotropy). The absorption frequency (the dispersion frequency of the complex permeability) decreases with about one order of magnitude, going from the calculated

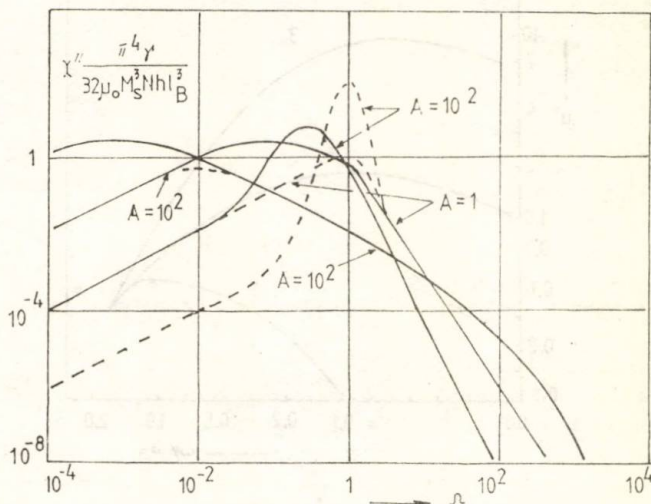


Fig. 4 — χ'' vs. frequency for three values of the relative damping A_1 , using (13).

value of 460 MHz to the measured value of 80 MHz for a 49.7 Fe₂O₃ - 50.3 NiO ferrite with a quasistatic initial permeability of 20...30 [2]. The sample is characterized by a high dispersion frequency and a low quality factor, given the small permeability and large loss resistance; the composition shows a deficit of Fe⁺² ions.

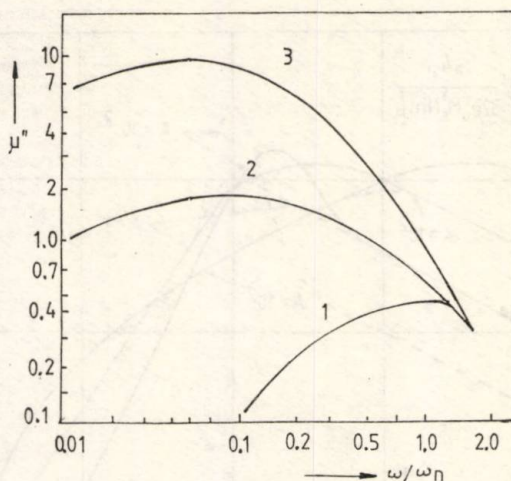
5. The difference between the values calculated when using the previous model and the 80 MHz as measured [2] is still too high. We consider that the Bloch wall exponential distribution given by the relation (13) does not reflect very well the real situation. As follows from this distribution, the number of the very short Bloch wall is very great. Yet, it is more reasonably to assume that there is a most probable length l_B , but both the number of the very long and very short Bloch walls is very small.

Considering the distribution of the 180° Bloch wall length under the from [3]

$$(15) \quad n = K l_p \exp \left[-\frac{l_p}{l_B} \right]$$

we find the complex permeability given in Fig. 5. The curve 1 was drawn considering the same length for all the walls, while curve 2 was represented for the length distribution given by (13). The curve 3 presents the complex permeability values calculated by means of the distribution (15).

The model that we suggested can be still improved. For instance, by replacing the linear variation of the Bloch wall number vs. their length, the position and width

Fig. 5 — μ'' vs. ω/ω_n .

of the dispersion maximum will modify.

Taking into account the 90° Bloch walls they will have a certain influence on the microscopical magnetization (relation (5)) and accordingly on the dispersion parameters.

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SPECTRE DE PERMEABILITATE LA FERITE

(Rezumat)

Se dezvoltă un model "membrană elastică" a pereților Bloch pentru calculul permeabilității magnetice complexe. Se determină frecvența de dispersie a permeabilității, ținând seama de diverse distribuții posibile ale lungimii pereților de domenii.

DECOMPOSITION OF LIQUID-VAPOUR EQUILIBRIUM CURVES OF SIGMOIDAL TYPE IN HYPERBOLIC TYPE COMPONENTS

BY

M. HABA and C.G. HABA

1. Introduction. Liquid-vapour equilibrium curves are frequently used to determine the activity coefficients of system components. For this reason, the knowledge as exactly as possible of the parameters of these curves has an outstanding importance.

In this paper we aimed to analyze in what measure complicated types of equilibrium curves can be expressed as linear combinations of equilibrium curves with simpler shapes. We focus specially on sigmoidal types, trying to model them using hyperbolic arcs.

Similar problems arise also in applications as processing experimental data related to spectra shifts in ternary solutions [1], [2] and ionic exchanges on zeolites [3].

2. Theory. 2.1. We consider a binary system with two phases, composed of components *I* and *II* which can simultaneously be in the *L* (liquid) and *V* (vapour) phases, and which are miscible in any proportion. By *II* we denote the more volatile component. In thermodynamics, the liquid-vapour equilibrium is characterized by relations [4]

$$(1)_a \quad \ln \frac{a_{II}^V}{a_{II}^L} = -(\Lambda_{II}^L + \Lambda_{II}^V)$$

and

$$(1)_b \quad \ln \frac{a_I^V}{a_I^L} = -(\Lambda_I^L + \Lambda_I^V),$$

where

$$(2) \quad a_k^w = \gamma_k^w x_k^w, \quad (k=I, II; w=V, L)$$

are the component activities in the two phases (defined using activity coefficients γ_k^w

and molar fractions x_k^w), and $\Lambda_K = \Lambda_{KT} + \Lambda_{KP}$ are normalized energetic quantities depending on temperature and pressure, but being independent of composition. If temperature is constant, then $\Lambda_{KT} = 0$ and the curves described by Eqs. (1)_{a,b} are isothermes; if pressure is constant, then $\Lambda_{KP} = 0$ and the before mentioned curves become isobars. In this paper we will especially deal with the isotherms.

Using the notations

$$(3)_a \quad A^w = \frac{a_{II}^w}{a_I^w}, \quad \Gamma^w = \frac{\gamma_{II}^w}{\gamma_I^w}, \quad X^w = \frac{x_{II}^w}{x_I^w},$$

and taking into account that, by definition,

$$(3)_b \quad a_{II}^w = 1 - a_I^w \equiv a^w \quad \text{and} \quad x_{II}^w = 1 - x_I^w \equiv x^w,$$

Eqs. (1)_{a,b} can be expressed as it follows:

$$(4)_{a,b} \quad X^V = B \Gamma X^L \quad \text{or} \quad x^V = \frac{x^L}{1 - K_B \Gamma (1 - x^L)},$$

where

$$(4)_c \quad B = \exp[-(\Lambda_{II} - \Lambda_I)], \quad \Gamma = \frac{\Gamma^L}{\Gamma^V} \quad \text{and} \quad K_B \Gamma = 1 - \frac{1}{B \Gamma}.$$

Here from follows that when the binary system is ideal both of its phases, i.e. $\Gamma^L = \Gamma^V = 1$, then

$$(5)_{a,b} \quad X^V = B X^L, \quad x^V = \frac{x^L}{1 - K_B (1 - x^L)}.$$

If the molar volume of the components in liquid phase is negligible with respect to their molar volume in vapour phase, then, at constant temperature, the partition constant B , named also relative volatility, and the screening factor K_B have the expressions

$$(5)_c \quad B = \frac{P_{II}^0}{P_I^0} \quad \text{and} \quad K_B = \frac{P_{II}^0 - P_I^0}{P_I^0},$$

where P_I^0 and P_{II}^0 are the vapour tensions of the components. In these conditions, if $P_I^0 = P_{II}^0$, then clearly $B = 1$ and $K_B = 0$.

Plots of Eqs. (5)_{a,b} for different values of parameters B or K_B are given in Figs. 1 *a, b*. As we can see, the first equation, plotted in logarithmic scales, represents a family of straight lines with slope equal to unit and intercept equal to $\ln B$. The second equation represents a family of equilateral hyperbola arcs placed above or below the first bisectrix as K_B is greater or smaller than zero. For $K_B = 0$, $x^V = x^L$, i.e., the corresponding arc of hyperbola coincides with the bisectrix. For $K_B = 1$, this arc is degenerated in the straight lines $x^L = 0$ and $x^V = 1$.

The situation is total different when $\Gamma^V = 1$, but Γ^L varies with the composition

X^L according to a certain law. Binary systems having an azeotropy point in which the two phases are identical, i.e. $X^V = X^L$ or $x^V = x^L$, are very frequent. In this case, Eqs. (4)_{a, b} correspond the curves given in Figs. 2 *a, b*. These curves can be described in two ways:

i) emphasizing their azeotropy point, i.e. expressing the variation of Γ^L as

$$\Gamma^L = C \exp P(X^L - X_a^L),$$

where C is a constant and P is a polynomial in $X^L - X_a^L$;

ii) considering that they are composed of simpler curves describing partial processes. In the rest of the paper we will refer to the second approach.

2.2. So, we suppose that $x^V \equiv y$ can be expressed in terms of $x^L \equiv x$ with a relation of the form

$$(6)_a \quad y = p_1 y_1 + p_2 y_2,$$

where every partial molar fraction of the vapour is

$$(6)_b \quad y_i = \frac{x}{1 - K_i(1 - x)} = \frac{B_i X}{1 + B_i X},$$

where $K_i = 1 - 1/B_i$ and p_i is the weight of y_i chosen to satisfy the equality

$$(6)_c \quad p_1 + p_2 = 1.$$

Under these circumstances, it can be proved that Eq. (4)_a can be written as follows

$$(7) \quad Y = X = \frac{p_1 B_1 + p_2 B_2 + B_1 B_2 X}{1 + (p_1 B_2 + p_2 B_1) X}.$$

We can easily see that for $X \rightarrow 0$,

$$(8)_a \quad B(0) \equiv B_0 = p_1 B_1 + p_2 B_2$$

and for $X \rightarrow \infty$,

$$(8)_b \quad \frac{1}{B(\infty)} \equiv \frac{1}{B_\infty} = \frac{p_1}{B_1} + \frac{p_2}{B_2}.$$

Using the notation

$$(8)_c \quad X^* = \frac{1}{p_1 B_2 + p_2 B_1},$$

the equivalent partition factor $B(X)$ can be written as

$$(9) \quad B(X) = \frac{B_0 X^* + B_\infty X}{X^* + X}.$$

A peculiar value of this function is $B(X^*) = (B_0 + B_\infty)/2$.

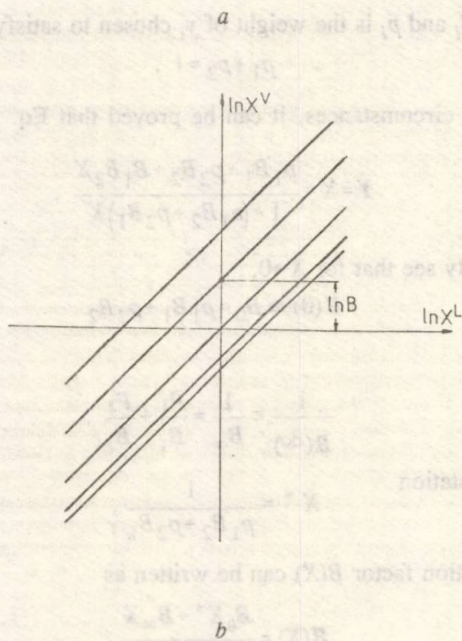
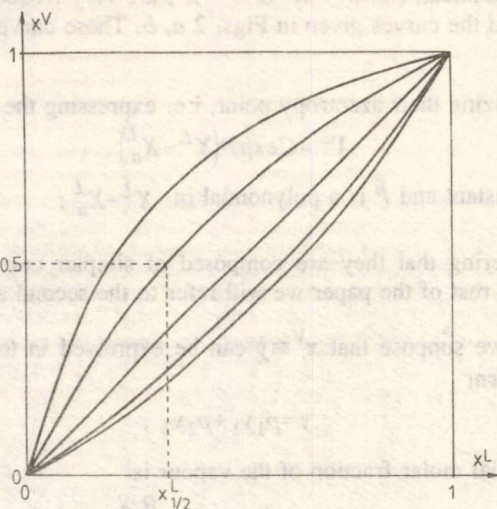
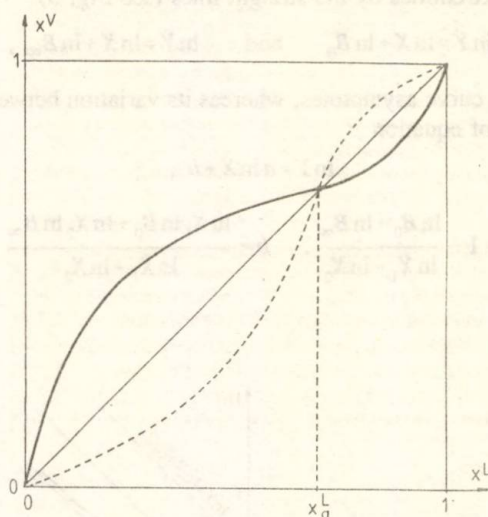
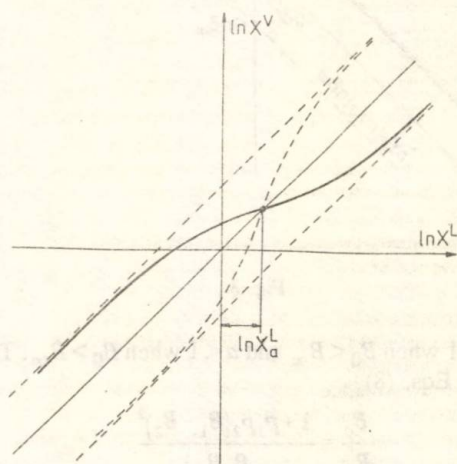


Fig. 1



a



b

Fig. 2

The above results indicate that the curve depicted in Fig. 2b can be approximated at the extremities by the straight lines (see Fig. 3)

$$(10)_{a,b} \quad \ln Y = \ln X + \ln B_0 \quad \text{and} \quad \ln Y = \ln X + \ln B_\infty,$$

which constitute the curve asymptotes, whereas its variation between the asymptotes, by the straight line of equation

$$(11)_a \quad \ln Y = a \ln X + b,$$

where

$$(11)_{b,c} \quad a = 1 - \frac{\ln B_0 - \ln B_\infty}{\ln X_1 - \ln X_2}, \quad b = \frac{\ln X_1 \ln B_0 + \ln X_2 \ln B_\infty}{\ln X_1 + \ln X_2}.$$

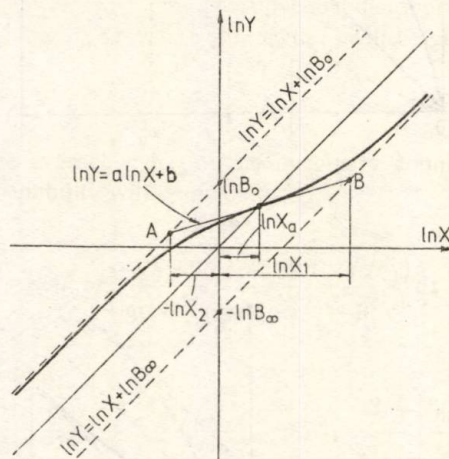


Fig. 3

We note that $a > 1$ when $B_0 < B_\infty$ and $a < 1$ when $B_0 > B_\infty$. Taking into account that in accordance with Eqs. (8)_{a,b,c}

$$(12) \quad \frac{B_0}{B_\infty} = \frac{1 + p_1 p_2 (B_1 - B_2)^2}{B_1 B_2}$$

and considering that $B_1 \neq B_2$, $a > 1$ only if $p_1 p_2 / B_1 B_2 < 0$, while $a < 1$ only if $p_1 p_2 / B_1 B_2 > 0$.

2.3. If we possess an appropriate number of experimental points, 15...30, then the parameters of the equilibrium curves of sigmoidal type can be computed as follows:

i) to determine the parameters B_0 , B_∞ and X^* we choose three points on the curve. Let these points be $P_i(Y_i, X_i)$, ($i=1, 2, 3$). Introducing in Eqs. (7) and (9) the coordinates of these points, we form a system of three equations with three unknown variables

$$(13) \quad \frac{B_0 X^* + B_\infty X_j - (X^* + X_j) Y_j}{X_j} = 0, \quad j=1, 2, 3.$$

Solving this system, we obtain

$$(14) \quad B_\infty = \frac{D_\infty}{D}, \quad X^* = \frac{D_x}{D} \quad \text{and} \quad B_0 = \frac{1}{X^*} \frac{(X^* + X_1) Y_1}{X_1},$$

where

$$(15)_{a,b,c} \quad D_\infty = c_{12} b_{13} - c_{13} b_{12}, \quad D_x = a_{12} c_{13} - a_{13} c_{12}, \quad D = a_{12} b_{13} - a_{13} b_{12};$$

$$(15)_{d,e} \quad a_{12} = X_1 - X_2, \quad b_{12} = \frac{Y_2}{X_2} - \frac{Y_1}{X_1}, \quad c_{12} = Y_1 - Y_2,$$

$$a_{13} = X_1 - X_3, \quad b_{13} = \frac{Y_3}{X_3} - \frac{Y_1}{X_1}, \quad c_{13} = Y_1 - Y_3.$$

ii) in the next step we compute the values for the parameters B_1 , B_2 and $p \equiv p_2 = 1 - p_1$ by using Eqs. (8)_{a,b,c}. The computing relations are

$$(16)_{a,b} \quad \Sigma \equiv B_1 + B_2 = \frac{B_0 + 1}{X^*}, \quad \pi \equiv B_1 B_2 = \frac{B_\infty}{X^*},$$

$$(16)_{c,d} \quad B_{1,2} = \frac{1}{2} \Sigma \pm (\Sigma^2 - 4\pi)^{1/2}, \quad p = \frac{B_0 - B_1}{B_2 - B_1}.$$

The parameters are then used to plot the theoretical curves which are compared with the experimental ones.

3. Comparison with Experimental Data. In order to verify the validity of the methodology presented above we have used vapour-liquid equilibrium data given in [5]. The binary systems for which have been carried out calculations as well as the obtained parameters are given in Table 1.

These parameters were calculated by us for 11 triplets of points chosen as follows:

Let $(X_1, Y_1), \dots, (X_n, Y_n)$ be the data points. One of the triplet points, the same for all triplets, is that for which $X = 1$. The other two are chosen symmetric to the first one, i.e. for the first triplet (X_1, Y_1) and (X_n, Y_n) , for the second triplet (X_2, Y_2)

and (X_{n-1}, Y_{n-1}) and so on.

With the obtained parameters we calculated the curves and computed the standard deviation for each triplet. We have then chosen the parameters B_1 , B_2 and p so that for the given triplet the standard deviation was the smallest.

Table 1
Parameters of vapour-liquid equilibrium curves

No	Binary system	B_1	B_2	p
1	Chloroform + Benzene	0.584	0.584	1.000
2	<i>n</i> -Hexane + Benzene	4.210	0.605	0.478
3	<i>n</i> -Heptane + Benzene	2.956	0.310	0.800
4	<i>n</i> -Octane + Benzene	3.031	0.184	0.959
5	Cyclohexane + Benzene	1.940	0.296	0.370
6	Carbon tetrachloride + Benzene	1.191	0.346	0.062
7	Acetone + Benzene	7.355	1.120	0.612
8	Acetaldehyde + Benzene	9.489	0.324	0.077
9	Benzene + Toluene (1 at)	2.246	1.063	-0.194
10	Benzene + Toluene (2 at)	1.814	0.188	-0.033
11	Benzene + Toluene (10 at)	1.592	0.044	-0.005
12	Benzene + Toluene (35 at)	1.397	0.077	0.013
13	<i>n</i> -Heptane + Toluene	3.963	0.804	0.626
14	<i>n</i> -Octane + Toluene	2.889	0.508	0.833
15	Cyclohexane + Toluene	3.442	0.440	0.106
16	Carbon tetrachloride + Toluene	2.239	2.239	1.000
17	Water + Acetic acid	13.227	1.403	0.037
18	Water + Propionic acid	5.885	0.035	0.059
19	Water + Butyric acid	14.080	0.004	0.036
20	Methanol + Water (1 at)	20.229	0.407	0.248
21	Methanol + Water (4.2 at)	6.927	0.598	0.236
22	Methanol + Water (11.16 at)	2.305	2.305	1.000
23	Ethanol + Water (0.13 at)	24.545	0.417	0.426
24	Ethanol + Water (0.66 at)	15.284	0.349	0.413
25	Ethanol + Water (1 at)	23.662	0.395	0.447
26	Iso-propanol + Water (1 at)	137.740	0.334	0.540
27	Butanol + Water (1 at)	55.600	0.110	0.450

As we can see in Figs. 4...6 there is a good agreement between the calculated and experimental data, especially in the cases when in the binary systems act universal intermolecular forces only, or when the components of the binary system are close as chemical nature. Where there are also specific molecular interactions, for instance hydrogen bonds (Fig. 5), we notice several deviations from the proposed models.

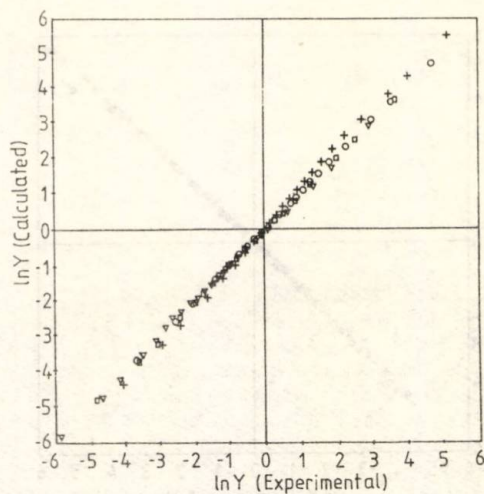


Fig. 4

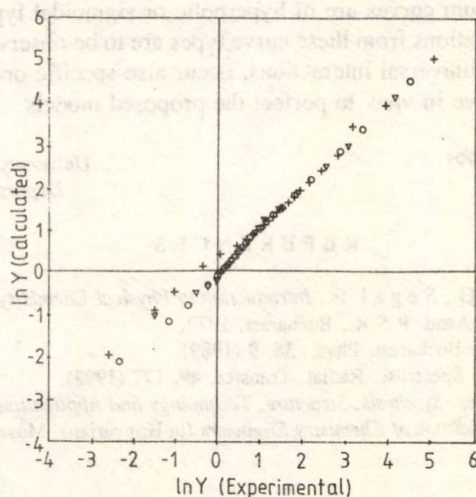


Fig. 5

4. Conclusions. 1. It was attempted to describe the vapour-liquid equilibrium curves by analytical expressions as simple as possible.

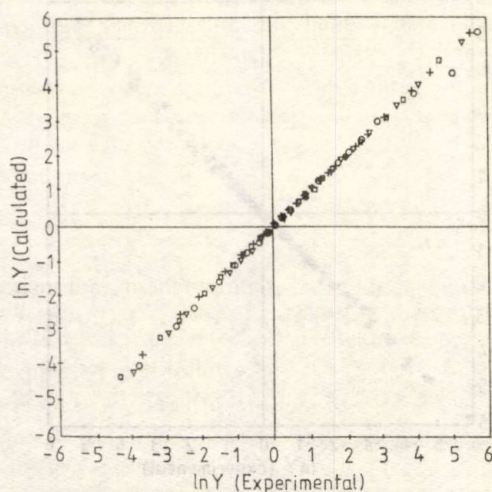


Fig. 6

2. It was remarked that there is a great number of binary systems for which the vapour-liquid equilibrium curves are of hyperbolic or sigmoidal type.

3. Notable deviations from these curve types are to be observed for the systems in which, besides the universal interactions, occur also specific ones. For describing these situations we have in view to perfect the proposed models.

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DESCOMPUNEREA CURBELOR DE ECHILIBRU LICHID-VAPORI DE TIP SIGMOIDAL ÎN COMPONENTE DE TIP HIPERBOLIC

(Rezumat)

Pentru determinarea coeficienților de activitate ai componentelor unor sisteme constituite din două sau mai multe substanțe se utilizează adesea curbele de echilibru lichid-vapori. Din acest motiv,

cunoașterea cât mai exactă a formei și parametrilor acestor curbe prezintă o importanță deosebită. Prezenta lucrare își propune să analizeze în ce măsură tipurile de curbe de echilibru mai complicate pot fi descrise prin combinații liniare ale unor curbe de echilibru de forme mai simple. S-au studiat în special curbele cu formă sigmoidală, care au fost modelate prin arce de hiperbolă. Această abordare s-a dovedit a fi valabilă, lucru demonstrat de buna concordanță între datele teoretice și cele experimentale.

RECENZII

BOOK REVIEWS

РЕЦЕНЗИИ

REINER HOPFER, **Informationsverarbeitung mit PASCAL**. Verlag Technik GmbH, Berlin, 1990, 331 S., ISBN 3-341-00522-6.

Die Programmiersprache PASCAL ist zweifellos eine der bedeutendsten höher Programmiersprachen. Sie erhielt ihren Namen zu Ehren des französischen Physikers und Mathematikers B. P a s c a l (1623 — 1662), der eine der ersten Rechmaschinen konstruierte. PASCAL wurde von N. W i r t h, Zürich, in den Jahren 1968 bis 1973 vor allem für die Ausbildung entwickelt. Anschliessend wurden entsprechende Programmiersysteme erprobt und längere Zeit Erfahrungen mit der Programmiersprache gesammelt. 1980 lag schliesslich ein erster Entwurf für einen internationalen Standard vor, den man 1981 als sog. ISO-PASCAL-Stand verabschiedete.

Aufgrund der Leistungsfähigkeit, Klarheit und leichten Erlernbarkeit der Sprache hat sich PASCAL schnell weltweit verbreitet und steht heute nahezu allen Rechnertypen — vom Mikrocomputer bis zur Grossrechenanlage — zur Verfügung. Weitere Vorteile von PASCAL sind durch die gute Lesbarkeit und Struktur der Programme sowie deren effective Übersetzung durch den Computer begründet. Die Programmiersprache PASCAL stellt relativ einfache sprachliche Ausdrucksmittel zur Verfügung, unterstützt die strukturierte Programmierung und gestattet den Aufbau vielfältiger problemorientierter Datenstrukturen. Sie vermittelt einen sauberen Programmierstil und ermöglicht ein systematisches Programmieren.

Mit dem vorliegenden Buch wird ein breiter Kreis von Lesern angesprochen. Das Hauptanliegen besteht darin, eine systematische Einführung in die Programmiersprache PASCAL und ihre Anwendung zu geben. Zunächst werden *grundlegende Kenntnisse über Aufbau und Programmierung eines Computers* vermittelt. Den wesentlichen Teil der Ausführungen bildet die *Beschreibung der Programmiersprache PASCAL unter Berücksichtigung des ISO-Standards*. Es sind alle Sprachelemente, Anweisungen und Datentypen detailliert enthalten und anhand von Syntaxdiagrammen und zahlreichen Beispielen erläutert. Ausserdem werden *wesentliche Sprachweiterungen* beschrieben, die in unterschiedliche und häufig genutzten PASCAL-Versionen vorkommen. Ausführlich wird auf *TURBO-PASCAL, Version 5.0*, eingegangen, da dieses Programmmentwicklungssystem besonders bei Mikrocomputern breite Verwendung findet. Darüber hinaus sind *praktische Hinweise zur Aufbereitung, Übersetzung und Ausführung eines PASCAL-Programms* unter der Steuerung des Betriebssystem UNIX enthalten. Im Anhang sind eine *Komprimierte Sprachbeschreibung von Standard-PASCAL* und *zusätzliche Sprachelemente von TURBO-PASCAL* aufgeführt.

Die im Buch benutzten Beispiele sowie deren Lösungen sind nach didaktische Gesichtspunkten ausgewählt. Besonderer Wert wird auf verständliche Algorithmen werden Struktogramme benutzt, die

den Entwurf gut strukturierter Programme unterstützen. Die Vermittlung des Stoffs ist so gewählt, dass man sich auch im Selbststudium die Programmiersprache PASCAL aneignen kann. Allerdings sollten dabei zusätzliche praktische Übungen an einem Computer möglich sein.

RICHARD FAISST, Musterpflichtenheft PPS- und Logistik-Systeme. Expert Verlag GmbH, Renningen, 1994, 317 S., ISBN 3-8169-0779-2.

Viele Probleme bei der Auswahl und Einführung von Anwendungssoftware lassen sich garantiert vermeiden, wenn der künftige EDV-Anwender vor der Auswahlentscheidung in diesem Musterpflichtenheft seine speziellen Anforderungen kennzeichnet.

Das Werk unterstützt den Leser in allen Phasen — von der Vorgehensweise, Leistungsfestlegung, Auswahlkriterien bis hin zur Vertragsgestaltung durch direkt übernehmbare Methoden und Praxishilfen. Das Musterheft hilft, aufgrund fundierter Daten die optimale Wahl zu treffen und trägt somit zur Sicherung des Unternehmens bei.

Inhalt: Anforderungen an EDV-Anwendungssysteme; Vorgehensweise und Auswahlkriterien für geeignete Anwendungssoftware; Wirtschaftlichkeitsberechnung; Bewertung der Software; Vertragsgestaltung; Systematische Einführung der Anwendersoftware; Organisatorische und personelle Voraussetzungen; Datenverarbeitungs-Umfeldbedingungen; Merkmale des Muster-Pflichtenheftes; Grunddaten/Stammdaten-Produktionsprogrammplanung/Absatzplanung; Bedarfsermittlung; Termin- und Kapazitätsplanung; Werkstattsteuerung; Bestandsführung; Bestellabwicklung; Rechnungsprüfung; Angebotsbearbeitung; Auftragsabwicklung; Kosten- und Leistungsrechnung; Schnittstellen.

PHILIPPE CLÉMENT and GÜNTER LUMER (Eds.), Evolution Equations, Control Theory and Biomathematics. Marcel Dekker, Inc., New York, 1994, 580 pp., ISBN 0-8247-8885-0.

Based on the *Third International Workshop-Conference on Evolution Equations, Control Theory, and Biomathematics* held recently in Han-sur-Lesse, Belgium, this timely reference provides in-depth coverage of the latest developments in a variety of complementary fields.

Examining important advances in evolution equations related to physical, engineering, and biological applications *Evolution Equations, Control Theory, and Biomathematics* addresses population models, epidemic models and diffusion models with shocks, control theory and optimal control theory and optimal control, Riccati, Hamilton-Jacobi, and Korteweg-de Vries equations, asymptotics of spectral functions, viscosity solutions, Laplace transform and perturbation methods, stochastic analysis, diffusion problems, and more!

With over 2600 display equations, bibliographic citations, and figures, *Evolution Equations, Control Theory, and Biomathematics* is an invaluable resource for pure and applied mathematicians; applied scientists in physics, engineering and mathematical biology; and upper-level undergraduate and graduate students involved in linear and nonlinear parabolic equations, evolutions equations, control theory, population dynamics, and mathematical biology.

From the *C o n t e n t s*: *Preface*; Contributors, Conference Program; Conference Participants; *Nonlinear and Dynamical Node Transition in Network. Diffusion Problems* (Joachim von B e l o w); *On the Balance Between Mutation and Frequency-Dependent Selection in Evolutionary Game Dynamics* (Immanuel M. B o m z e, Reinhard B ü r g e r); *Uniform Stabilization for the Solutions to a von Kármán Plate* (M. E. B r a d l e y, I. L a s i e c k a); *Viscosity Solutions and Partial Differential Inequalities* (Ovidiu C â r j ă, Corneliu U r s e s c u); *On a Class of C-Regularized Semigroups* (Ioana C i o r â n e s c u); *Hamilton-Jacobi Equations in Infinite Dimensions* (M. G. C r a n d a l, P. L. L i o n s); *Boundary Control Problem for Age-Dependent Equations*

(Giusepppe Da Prato, Mimmo Iannelli); *Simultaneous Well-Posedness* (Ralph de Laubenfels); *A Hilbert-Schmidt Property of Resolvent Differences of Singularly Perturbed Generalized Schrödinger Operators* (M. Demuth, J. A. van Casteren); *The "Cumulative" Formulation of Structured Population Models* (O. Diekmann); *...Smooth Solutions of Nonlinear Elliptic Systems with Dynamic Boundary Conditions* (Joachim Escher); *... On Second Order Implicit Differential Equations in Banach Spaces* (A. Favini, A. Yagi); *...Commuting Multiplicative Perturbations* (Albrecht Holderrieth); *Derivative Nonlinear Schrödinger Equations* (Horst Lange); *The Wave Equation Method in the Spectral Theory of Self-Adjoint Differential Operators* (B. M. Levitan); *Nonlinear Boundary Value Problems for a Class of Hyperbolic Systems* (Rodica Luca, Gheorghe Moroșanu); *Idempotent Semimoduli in Optimization Problems* (V. P. Maslov); *... Hyperbolic Singular Perturbation of a Quasilinear Parabolic Equation* (B. Najm); *...Rational Approximations of Analytic Semigroups* (S. Piskarev); *...On the Cauchy-Dirichlet Problem for an Equation of Third Order* (J. Popiotek); *Generalized Spectral Decomposition of the r -adic Renyi Map* (S. Tasaki, I. Antoniu); *Can We Understand What Life Is?* (René Thom); *On a Problem from Thermal Convection* (Wolf von Wahl).

D. S. MITRINOVIČ, J. E. PECARIČ and A. M. FINK, *Classical and New Inequalities in Analysis*. Kluwer Academic Publ., Dordrecht, The Netherlands, 1993, 740 pp., ISBN 0-7923-2064-6.

This volume presents a comprehensive compendium of classical and new inequalities as well as some recent extensions to well-known ones. Variations of inequalities ascribed to Abel, Jensen, Cauchy, Chebyshev, Hölder, Minkowski, Steffensen, Gram, Fejér, Jackson, Hardy, Littlewood, Polya, Schwarz, Hadamard and a host of others can be found in this volume. The more than 1200 cited references include many from the last 10 years which appear in a book for the first time.

The 30 chapters are all devoted to inequalities associated with a given classical inequality, or give methods for the derivation of new inequalities. Anyone interested in inequalities, from student to professional, will find their favorite inequality and much more.

Contents: *Preface; Organization of the Book, Notations; Convex functions and Jensen's Inequality; Some recent results involving means; Bernoulli's inequality; Cauchy's and related inequalities; Hölder's and Minkowski inequalities; Generalized Hölder and Minkowski inequalities; Connections between general inequalities; Chebyshev's inequality; Grüss' inequality; Abel's and related inequalities; Some inequalities for monotone functions; Young's inequality; Bessel's inequality; Cyclic inequalities; Triangle inequalities; Norm inequalities; More on norm inequalities; Gram's inequality; Fejér-Jackson's inequalities and related results; Mathieu's inequality; Shannon's inequality; Turian's inequality from the power sum theory; Continued fractions and Padé approximation method; Quasilinearization methods (for proving inequalities); The centroid method in inequalities; Dynamic programming and functional equation approaches to inequalities; Interpolation inequalities; Convex Mini-max inequalities-equalities; Name Index.*

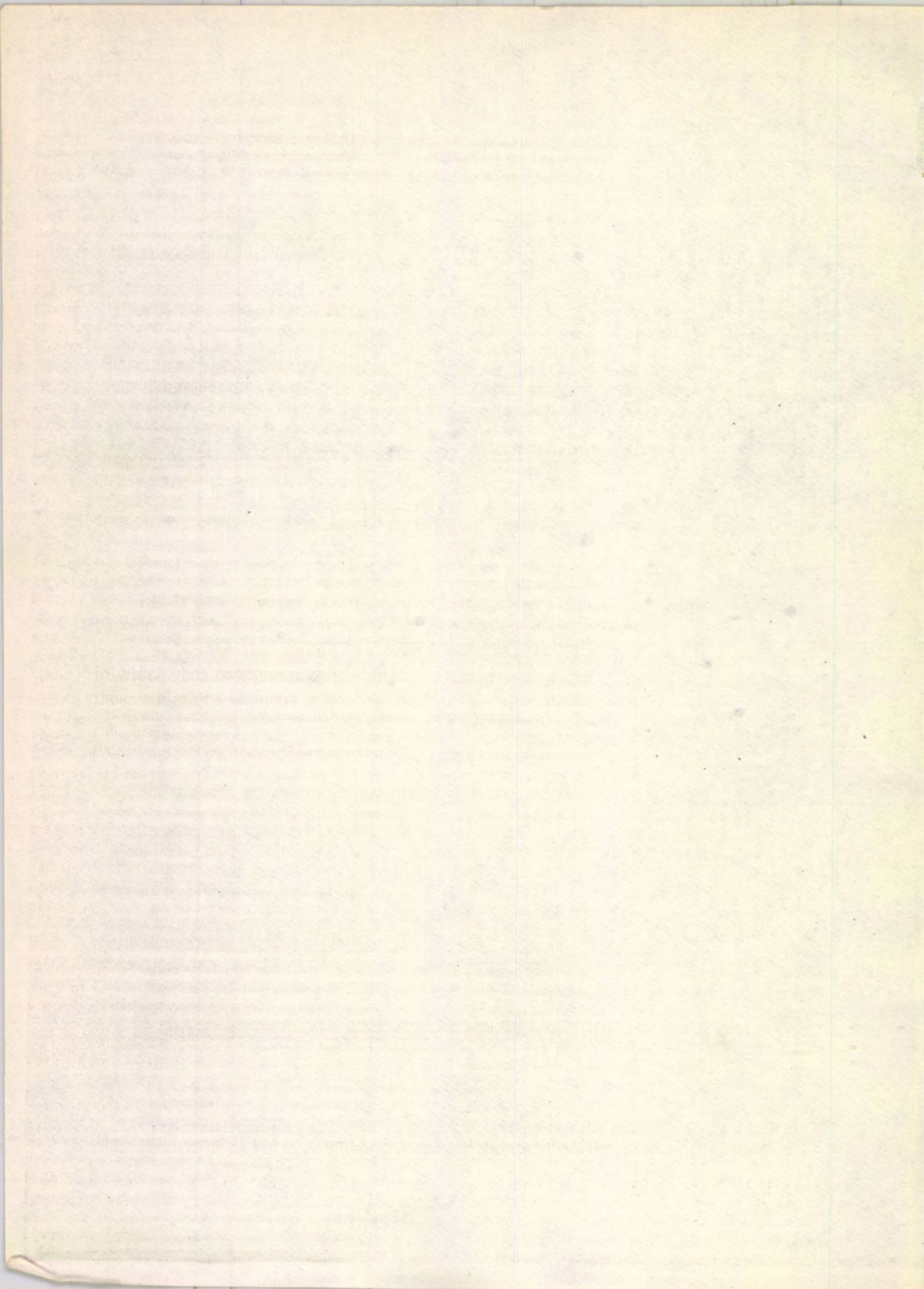
SEPPO HEIKKILÄ and V. LAKSHMIKANTHAM, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*. Marcel Dekker, Inc., New York, 1994, 514 pp., ISBN 0-8247-9224-6.

Providing the theoretical framework to model phenomena with discontinuous changes, this unique reference presents a *generalized monotone iterative method* in terms of upper and lower solutions appropriate for the study of discontinuous nonlinear differential equations and applies this method to *derive suitable fixed point theorems* in ordered abstract spaces.

Detailling the basic concepts behind a **generalized** monotone iterative method, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations* develops new existence and comparison results, extends the method of upper and lower solutions and the monotone iterative technique to systems in finite as well as infinite dimensional spaces, covers the existence and comparison of strong, weak, or mild solutions to discontinuous differential equations in Banach spaces without requiring any compactness hypotheses, treats first order partial differential equations and other important topics.

Contents: Preface; 1. *Methods and Prerequisites*; 2. *First Order Differential Equations*; 3. *Second Order Differential Equations*; 4. *Second Order Partial Differential Equations*; 5. *Differential Equations in Banach Spaces*; References; Index.





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BINARY TRIPLE ALGEBRAS

BY

ILIE BURDUJAN

0. Introduction. The *binary triple algebras* have been introduced by M. A. A k i v i s [1], in connection with the local Lie loops, who named them *W-algebras*. These algebras have been later treated by K.H. H o f m a n n and K. S t r a m b a c h [7], [8] as "tangent algebras" of local Lie loops; they termed them *Akivis-algebras*. Both these names are related to the theory of local Lie loops. But such algebras do not give a local characterization of local Lie loops. Indeed, it was proved in [3] that the tangent space at the identity element of a local Lie loop may be organized as a homogeneous system as considered by K. Y a m a g u t i (see [10]), the construction used for this being compatible with those given by the above mentioned authors.

On another hand, these algebras arise in the study of nonassociativity of binary algebras, and they also are generalizations of Lie triple algebras [9]. That is why it seems to be appropriate to name such algebras *binary triple algebras*. Moreover, as it has been proved in [4], a homogeneous system of K. Y a m a g u t i may be associated with any binary algebra. For certain binary algebras (for example see [6]), only the binary and ternary operations of the corresponding homogeneous systems determine the other ones. In this case, the two operations endow the underlying space of the algebra with a binary triple algebra structure, what justifies the interest in the study of such an algebraic structure.

The present paper is mainly concerned with *representations and extensions* of binary triple algebras.

1. Definitions. Examples. Throughout this paper, K will denote a commutative ring with unit (identity) element. The algebraic structures we are dealing with (abbreviated *BT-algebras*) are introduced by

Definition 1. A *binary triple algebra* over K is a K -module A with a bilinear operation $A \times A \rightarrow A$, $(x, y) \rightarrow [x, y]$ and a trilinear operation $A \times A \times A \rightarrow$

$\rightarrow A, (x, y, z) \rightarrow \langle x, y, z \rangle$, satisfying the following axioms:

$$(A_1) \quad [x, x] = 0 \quad \text{for} \quad \forall x \in A;$$

$$(A_2) \quad \langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle - \langle x, z, y \rangle - \langle z, y, x \rangle - \langle y, x, z \rangle = \\ = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \quad \text{for} \quad \forall x, y, z \in A.$$

Remark 1. It is well known that the axiom (A_1) implies the anticommutativity of the binary operation, i.e.

$$(1) \quad [x, y] = -[y, x] \quad \text{for} \quad \forall x, y \in A.$$

Conversely, Eq. (1) implies (A_1) only if K is not of characteristic 2.

Example 1. Let $Q(\cdot)$ be a n -dimensional analytic loop with a base point e , $A = T_e Q$ the tangent vector space at e , (U, ψ) — a local chart such that $e \in U$ and $\psi(e) = 0 \in \mathbb{R}^n$. If $x, y, z = x \cdot y \in U$ and $\psi(x) = (x^1, \dots, x^n)$, $\psi(y) = (y^1, \dots, y^n)$, $\psi(z) = (z^1, \dots, z^n)$ then $z^i = \phi^i(x^1, \dots, x^n; y^1, \dots, y^n)$ ($i = 1, 2, \dots, n$) will denote the local expressions of loop operation in chart (U, ψ) . Then, there exist the MacLaurin expansions

$$\phi^i(x, y) - \phi^i(y, x) = C_{jk}^i x^j y^k + T_{>2}(x, y), \quad (i = \overline{1, n}),$$

$$\phi^i(x, \phi(y, z)) - \phi^i(\phi(x, y), z) = t_{jkl}^i x^j y^k z^l + T_{>3}(x, y, z) \quad (i = \overline{1, n}),$$

where $T_{>2}(x, y)$ and $T_{>3}(x, y, z)$ are series of polynomials of degree > 2 and > 3 , respectively. If $\{e_1, \dots, e_n\}$ is a basis of A then A , endowed with the operations

$$[a, b] = C_{jk}^i a^j b^k e_i, \quad \langle a, b, c \rangle = t_{jkl}^i a^j b^k c^l e_i, \quad \forall a, b, c \in A$$

(where $a = a^i e_i$, $b = b^i e_i$, $c = c^i e_i$), becomes a real BT -algebra. When $Q(\cdot)$ is an analytic group, the BT -algebra A reduces to the classical Lie algebra.

This is a very important example because it alone legitimates the whole study of the binary triple algebras.

Example 2. Any K -module A may be considered as a BT -algebra if it is endowed with the operations

$$(2) \quad [x, y] = 0, \quad \langle x, y, z \rangle = 0, \quad \forall x, y, z \in A.$$

Example 3. Let $A(\cdot)$ be a nonassociative binary algebra over K . The operations

$$(3) \quad [x, y] = x \cdot y - y \cdot x, \quad \forall x, y \in A;$$

$$(4) \quad \langle x, y, z \rangle = x \cdot (y \cdot z) - (x \cdot y) \cdot z, \quad \forall x, y, z \in A$$

endow A with a BT -algebra structure.

Remark 2. If K is a field whose characteristic is neither 2 nor 4, then a more natural BT -algebra structure on a nonassociative binary algebra $A(\cdot)$ is given by

$$(5) \quad [x, y] = \frac{1}{2}(x \cdot y - y \cdot x), \quad \forall x, y \in A;$$

$$(6) \quad \langle x, y, z \rangle = \frac{1}{4}(x \cdot (y \cdot z) - (x \cdot y) \cdot z), \quad \forall x, y, z \in A.$$

Example 4. Anyone of the following kinds of algebras, namely Lie-algebras, Lie triple systems, general Lie triple systems, Malcev algebras, may be — in a natural way — equipped with *BT*-algebra structures.

Definition 2. The *BT*-algebra A is called a *commutative BT*-algebra if

$$(7) \quad [x, y] = 0, \quad \forall x, y \in A;$$

$$(8) \quad \langle x, y, z \rangle = -\langle z, y, x \rangle, \quad \forall x, y, z \in A.$$

Example 5. The *BT*-algebra associated with a commutative (local) analytic loop is a commutative one.

Example 6. The *BT*-algebra associated with a binary (nonassociative) K -algebra $A(\cdot)$ as in Example 3 (or Remark 2) is a commutative one if and only if $A(\cdot)$ is a commutative algebra.

Definition 3. The elements x, y, z of a *BT*-algebra A are called *associative elements* iff $\langle x, y, z \rangle = 0$.

Example 7. If G is a Lie subgroup of a (local) analytic loop Q , then any three elements of $T_e G \subset T_e Q$ will be associative elements of the *BT*-algebra $T_e Q$ (see Example 1).

Definition 4. The set $C(A) = \{a \in A : [a, x] = 0, \langle a, x, y \rangle = \langle x, a, y \rangle = \langle x, y, a \rangle = 0, \forall x, y \in A\}$ is called the *center* of the *BT*-algebra A .

Definition 5. A is called an *Abelian BT*-algebra if $[x, y] = 0$ and $\langle x, y, z \rangle = 0$ for any $x, y, z \in A$ (i.e., $C(A) \equiv A$).

Proposition 1. $C(A)$ is an *Abelian BT*-subalgebra of A .

Example 8. If the *BT*-algebra A is derived from the binary algebra $A(\cdot)$, then its center is just the center of $A(\cdot)$.

Example 9. The *BT*-algebra associated to a commutative associative binary algebra is both an *Abelian* and a commutative *BT*-algebra.

2. Representations. Let us consider two *BT*-algebras A and B , the linear mappings

$$(9) \quad \phi : A \rightarrow L_1(B, B), \quad \phi_i : A \rightarrow L_2(B, B) \quad (i = 1, 2, 3)$$

(when, for any $i \in \mathbb{N}^*$, $L_i(B, B)$ is the set of all B -valued i -multilinear mappings on B), and the bilinear mappings

$$(10) \quad \phi_{ij} : A \times A \rightarrow L_1(B, B) \quad (1 \leq i < j \leq 3).$$

The complete definitions of the mappings involved in (9) and (10) will be given later (when it will be necessary, only).

Then, the mappings $\phi^i: A \rightarrow L_2(B, B)$ may be built up (for $i = 1, 2, 3$) by

$$(11) \quad \phi^1(x)(y, z) = \underset{\text{not}}{\phi_x([y, z]_B)}, \quad \phi^2(x)(y, z) = \underset{\text{not}}{[\phi_x(z), y]_B},$$

$$(12) \quad \phi^3(x)(y, z) = \underset{\text{not}}{[z, \phi_x(y)]_B}, \quad \forall x \in A, \forall y, z \in B,$$

(here $[x, y]_B$ denotes the binary multiplication on B , and $\phi_x = \phi(x)$). Let us also define the linear mapping $\Phi: A \rightarrow L_2(B, B)$ by

$$(13) \quad \Phi(x) = \phi_1(x) - \phi_2(x) + \phi_3(x) - \frac{1}{2}[\phi^1(x) + \phi^2(x) + \phi^3(x)]$$

for $\forall x \in A$, and let us put

$$(14) \quad \phi_x * \phi_y = \phi_x \circ \phi_y - \phi_{12}(x, y) + \phi_{13}(x, y) - \phi_{23}(x, y), \quad \forall x, y \in A;$$

$$(15) \quad [\phi_x, \phi_y]^* = \phi_x * \phi_y - \phi_y * \phi_x, \quad \forall x, y \in A.$$

Finally, let us consider the function $s: B \times B \rightarrow B \times B$

$$(16) \quad s(x, y) = (y, x), \quad \forall (x, y) \in B \times B.$$

Definition 6. A representation R of the BT -algebra A in the BT -algebra B consists of seven linear functions $(\phi, \phi_i, \phi_{ij})$ ($i, j = 1, 2, 3; i < j$) of type (9) and (10) satisfying the conditions

$$(17) \quad [\phi_x, \phi_y]^* = \phi_{[x, y]}, \quad \forall x, y \in A;$$

$$(18) \quad \Phi_x = \Phi_x \circ s, \quad \forall x \in A.$$

Let us denote by $\mathfrak{R}(A, B)$, the set of all representations of the BT -algebra A in the BT -algebra B .

For the BT -algebra A let us define the mappings:

$$\phi: A \rightarrow L_1(A, A), \quad \phi(x)(y) = [x, y]_A = \phi_x(y),$$

$$\phi_1: A \rightarrow L_2(A, A), \quad \phi_1(x)(y, z) = \langle x, y, z \rangle_A,$$

$$\phi_2: A \rightarrow L_2(A, A), \quad \phi_2(x)(y, z) = \langle y, x, z \rangle_A,$$

$$(19) \quad \phi_3: A \rightarrow L_2(A, A), \quad \phi_3(x)(y, z) = \langle y, z, x \rangle_A,$$

$$\phi_{12}: A \times A \rightarrow L_1(A, A), \quad \phi_{12}(x, y)(z) = \langle x, y, z \rangle_A,$$

$$\phi_{13}: A \times A \rightarrow L_1(A, A), \quad \phi_{13}(x, y)(z) = \langle x, z, y \rangle_A,$$

$$\phi_{23}: A \times A \rightarrow L_1(A, A), \quad \phi_{23}(x, y)(z) = \langle z, x, y \rangle_A,$$

for any $x, y, z \in A$. From (18) and Definition 6 we get

Proposition 2. The functions $(\phi, \phi_i, \phi_{ij})$ ($i, j = 1, 2, 3; i < j$) defined by (18) realize a representation of the BT-algebra A in itself.

Definition 7. The representation of A in itself defined by (18) will be called the *adjoint representation* of BT-algebra A in itself and it will be denoted by Ad_A .

If B is a K -module viewed as an Abelian BT-algebra (see the Example 2), then a representation of a BT-algebra A in B may be defined, too. Such a representation consists from a set of mappings $(\phi, \phi_i, \phi_{ij})$ ($i, j = 1, 2, 3; i < j$) defined as before. Obviously, in this case we have

$$\Phi = \phi_1 - \phi_2 + \phi_3.$$

3. Ideals. Let A be a BT-algebra over the ring K .

Definition 8. Any K -submodule B of A satisfying

(19) $[B, A] \subset B, \langle B, A, A \rangle \subset B, \langle A, B, A \rangle \subset B, \langle A, A, B \rangle \subset B$
is called an *ideal* of A .

Remark 3. Any ideal B of A is invariant for the representation Ad_A . Indeed, there exists

$$\phi(x)(B) \subset B, \phi_i(x)(B \times B) \subset B, \phi_{ij}(x, y)(B) \subset B, \forall x, y \in A,$$

where $(\phi, \phi_i, \phi_{ij})$ ($i, j = 1, 2, 3; i < j$) are given by (18). Moreover, this is a characteristic property of the BT-algebra ideals. Obviously, Ad_A induces on B the representation Ad_B and $\text{Ad}_A \in \mathfrak{R}(A, B)$.

Example 10. The kernel of any BT-morphism is an ideal.

Definition 9. A BT-subalgebra B of A is called *characteristic ideal* of A if it is invariant for any $R \in \mathfrak{R}(A, A)$.

Proposition 3. a) If B is an ideal (resp. characteristic ideal) of A and C is a characteristic ideal of B , then C is an ideal (resp. a characteristic ideal) of A .

b) If B and C are ideals (resp. characteristic ideals) of A , then $B + C$ and $B \cap C$ are ideals (resp. characteristic ideals) of A .

Proposition 4. The center of any BT-algebra is an ideal.

4. Extensions of BT-algebras. Let A, B and E be BT-algebras over K .

Definition 10. Any exact sequence of BT-algebras

$$(E) \quad 0 \longrightarrow A \xrightarrow{\rho} E \xrightarrow{\sigma} B \longrightarrow 0$$

where ρ and σ are BT-algebra morphisms and 0 is the trivial BT-algebra, is called an *extension* of B by A .

In this case, $N = \ker \sigma$ is an ideal called the *extension kernel*. The morphism ρ is an BT-isomorphism between A and N , while σ induces a BT-isomorphism between E/N and B . It is a custom to say that E is an extension of B by A .

Definition 11. Two extensions

$$(\mathcal{E}_1) \quad 0 \longrightarrow A \xrightarrow{\rho_1} E_1 \xrightarrow{\sigma_1} B \longrightarrow 0$$

$$(\mathcal{E}_2) \quad 0 \longrightarrow A \xrightarrow{\rho_2} E_2 \xrightarrow{\sigma_2} B \longrightarrow 0$$

are called *equivalent* if there exists the BT-morphism $\tau: E_1 \rightarrow E_2$ such that the diagram

$$\begin{array}{ccccc} & & E_1 & & \\ & \nearrow & \downarrow \tau & \nwarrow & \\ A & & & & B \\ & \nwarrow & \downarrow \tau & \nearrow & \\ & & E_2 & & \end{array}$$

(Note: The diagram in the image is a diamond shape with A on the left, B on the right, E1 at the top, and E2 at the bottom. Arrows are: A to E1 labeled ρ1, A to E2 labeled ρ2, E1 to B labeled σ1, E2 to B labeled σ2, and E1 to E2 labeled τ.)

is commutative.

Proposition 5. If the two extensions $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ are equivalent, then the BT-morphism τ (see Definition 11) is a BT-isomorphism.

Let $\mathcal{F}(A, B)$ denote the set of all extensions of B by A . On $\mathcal{F}(A, B)$ it may be defined the binary relation $\mathfrak{R}$ by $\mathcal{E}_1 \mathfrak{R} \mathcal{E}_2$ if and only if $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent (via Definition 11).

Proposition 6. The binary relation $\mathfrak{R}$ is an equivalence relation on $\mathcal{F}(A, B)$.

The necessity of introducing the notion of section of an extension is emphasized by the following

Proposition 7. Let

$$(\mathcal{E}) \quad 0 \longrightarrow A \xrightarrow{\rho} E \xrightarrow{\sigma} B \longrightarrow 0$$

be an extension of the BT-algebra B by the BT-algebra A and N its kernel.

a) If there exists a BT-subalgebra M of E that is a supplement of N in the K -module E , then σ induces a BT-isomorphism σ_M between M and B ; the inverse K -linear morphism θ of σ_M is a BT-morphism of B in E and $\sigma \circ \theta = \text{id}_B$.

b) Conversely, if there exists a BT-morphism $\theta: B \rightarrow E$ such that $\sigma \circ \theta = \text{id}_B$, then $\theta(B)$ is a BT-algebra of E supplementary to N .

This Proposition allows us to give the following

Definition 12. Any K -linear mapping $\theta: B \rightarrow E$ satisfying $\sigma \circ \theta = \text{id}_B$ is called a *section* of the extension

$$(\mathcal{E}) \quad 0 \longrightarrow A \xrightarrow{\rho} E \xrightarrow{\sigma} B \longrightarrow 0.$$

Definition 13. The extension $(\mathcal{E})$ is called:

1. *modularly-split* if there exists a section θ of $(\mathcal{E})$;
2. *split* (or *inessential*) if there exists a section θ of $(\mathcal{E})$ which is a BT-morphism;
3. *central* if $\ker \sigma$ is contained in the center of E ;
4. *Abelian* if $\ker \sigma$ is an Abelian ideal of E ;

5. *trivial* if there exists a section for which the image is an ideal of E .

Remark 4. If E has a finite dimension, then $(\mathcal{E})$ is always a modularly-split extension.

Remark 5. Any central extension is an Abelian one.

Proposition 8. *If the extension $(\mathcal{E})$ is a trivial one with the kernel N and M is a supplementary ideal of N , then $E = N \oplus M$ and E may be isomorphically identified with $N \times M$, the Cartesian product of BT-algebras N and M .*

Proposition 9. *Any central and split extension is a trivial one.*

5. Semi-direct Product of BT-algebras. Let A and B be two BT-algebras over K . As well as for Lie algebras, the problem of the determination of all extensions of B by A appeared. This problem is very difficult to be solved, but a partial answer will be obtained, namely: "the equivalence classes of split extensions of B by A are in 1-1 correspondence with the semi-direct products of B by A (see Definition 14)".

Let

$$(\mathcal{E}) \quad 0 \longrightarrow A \xrightarrow{\rho} E \xrightarrow{\sigma} B \longrightarrow 0$$

be a split extension of B by A . Since ρ is a BT-monomorphism, A will be identified with $\rho(A)$; the map θ being a BT-monomorphism, B and $\theta(B)$ will be identified, too. So, the K -module E may be identified with the K -module $A \times B$. Since E is a split extension of B by A and A is identified with an ideal of B , the adjoint representation Ad_E induces a representation of B in A .

Proposition 10. *Two equivalent split extensions of B by A induce the same representation of B in A (via Ad_E).*

Let A and B be two BT-algebras over K and $R = (\phi, \phi_i, \phi_{ij}) \in \mathfrak{R}(B, A)$. Let us define on $E = A \times B$ the multilinear operations:

$$(20) \quad \underset{\text{def}}{[(a, b), (a', b')]} = ([a, a'] + \phi(b)(a') - \phi(b')(a), [b, b']),$$

$$(21) \quad \begin{aligned} &\underset{\text{def}}{\langle (a, b), (a', b'), (a'', b'') \rangle} = \langle a, a', a'' \rangle + \phi_1(b)(a', a'') + \\ &+ \phi_2(b')(a, a'') + \phi_3(b'')(a, a') + \phi_{12}(b, b')(a'') + \phi_{13}(b, b'')(a') + \\ &+ \phi_{23}(b', b'')(a), \langle b, b', b'' \rangle \end{aligned}$$

$$\forall (a, b), (a', b'), (a'', b'') \in A \times B.$$

Proposition 11. *The K -module $E = A \times B$, endowed with the two operations defined by (20) and (21), is a BT-algebra over K .*

Definition 14. The BT-algebra $E = A \times B$, earlier defined by (20), (21), is called the *semi-direct product* of the BT-algebra B by the BT-algebra A , corresponding to the representation $R \times \mathfrak{R}(B, A)$ and it will be denoted by $A \times_R B$.

Proposition 12. *The sequence*

$$0 \longrightarrow A \xrightarrow{\rho} A \times_R B \xrightarrow{\sigma} B \longrightarrow 0$$

where $\rho(a) = (a, 0)$, $\forall a \in A$ and $\sigma(a, b) = b$, $\forall (a, b) \in A \times B$, is an exact sequence and it also is a split extension of B by A having the property that $\text{Ad}_A \times_R B$ induces the representation R of B in A .

It may also be proved the

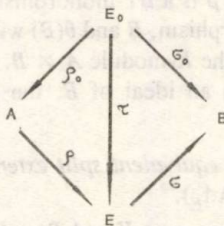
Proposition 13. *Let A and B be two BT-algebras over K and*

$$(E) \quad 0 \longrightarrow A \xrightarrow{\rho} E \xrightarrow{\sigma} B \longrightarrow 0$$

be a split extension of B by A , θ be a BT-isomorphism between B and a BT-subalgebra of E such that $\sigma \circ \theta = \text{id}_B$ and R be the representation of B by A induced by Ad_E . If

$$(E_0) \quad 0 \longrightarrow A \xrightarrow{\rho_0} E_0 \xrightarrow{\sigma_0} B \longrightarrow 0$$

is the split extension of B by A built up via semi-direct product of B and A , corresponding to the representation R , then the mapping $(a, b) \mapsto \rho(a) + \theta(b)$ is a BT-isomorphism between E_0 and E such that the following diagram is a commutative one



Combining the results of Propositions 10 – 13, the following main result may be stated:

Theorem 1. *There exists a 1-1 correspondence between the equivalence classes of split extensions of the BT-algebra B by the BT-algebra A and the elements of the set $\mathcal{R}(B, A)$.*

Remark 6. Any split extension of B by A is, up a BT-isomorphism, a semi-direct product of B by A , corresponding to a certain representation $R \in \mathcal{R}(B, A)$.

6. Abelian Extensions of BT-algebras. Let

$$(E) \quad 0 \longrightarrow A \xrightarrow{\rho} E \xrightarrow{\sigma} B \longrightarrow 0$$

be an Abelian extension of B by A . In this case, the ideal $\rho(A)$ of E uses the K -module structure induced by that of the K -module A , only. To any such an extension we may associate a representation of B in A induced by Ad_E . Indeed, let $b_1, b_2 \in B$ be two arbitrarily choosen elements. Then there exists the elements $e_1, e_2 \in E$ such that $\sigma(e_1) = b_1$, $\sigma(e_2) = b_2$. Let us now define the functions

$$\phi: B \rightarrow L_1(A, A), \quad \phi_i: L_2(A, A), \quad \phi_{ij}: A \times A \rightarrow L_1(A, A) \quad (i, j = 1, 2, 3; i < j)$$

by

$$(22.1) \quad \phi(b_1)(a) = \rho^{-1}([e_1, \rho(a)]),$$

$$(22.2) \quad \phi_1(b_1)(a, a') = \rho^{-1}(\langle e_1, \rho(a), \rho(a') \rangle),$$

$$(22.3) \quad \phi_2(b_1)(a, a') = \rho^{-1}(\langle \rho(a), e_1, \rho(a') \rangle),$$

$$(22.4) \quad \phi_3(b_1)(a, a') = \rho^{-1}(\langle \rho(a), \rho(a'), e_1 \rangle),$$

$$(22.5) \quad \phi_{12}(b_1, b_2)(a) = \rho^{-1}(\langle e_1, e_2, \rho(a) \rangle),$$

$$(22.6) \quad \phi_{13}(b_1, b_2)(a) = \rho^{-1}(\langle e_1, \rho(a), e_2 \rangle),$$

$$(22.7) \quad \phi_{23}(b_1, b_2)(a) = \rho^{-1}(\langle \rho(a), e_1, e_2 \rangle),$$

$$\forall a, a' \in A, \quad \forall b_1, b_2 \in B.$$

Remark 7. Since $[A, A] = 0$, $\langle A, A, A \rangle = 0$, $(\phi, \phi_i, \phi_{ij})$ ($i, j = 1, 2, 3; i < j$) are well defined by (22.1 – 7).

Proposition 14. The functions $(\phi, \phi_i, \phi_{ij})$ ($i, j = 1, 2, 3; i < j$) defined by (22.1 – 7) give a representation of the BT-algebra B in the K -module A .

Proposition 15. Two equivalent Abelian extensions of the BT-algebra B by the K -module A induce the same representation of B in A (via (22.1 – 7)).

Throughout in the sequel modularly-split extensions will be considered, only. Let

$$(E) \quad 0 \rightarrow A \xrightarrow{\rho} E \xrightarrow{\sigma} B \rightarrow 0$$

be a modularly-split Abelian extension. Then a section $\theta: B \rightarrow E$ may be considered. In relation to this section, two new functions $f: B \times B \rightarrow E$, $g: B \times B \times B \rightarrow E$ will be defined by

$$(23) \quad f(b_1, b_2) = [\theta(b_1), \theta(b_2)]_E - \theta([b_1, b_2]_B),$$

$$(24) \quad g(b_1, b_2, b_3) = \langle \theta(b_1), \theta(b_2), \theta(b_3) \rangle - \theta(\langle b_1, b_2, b_3 \rangle_B),$$

$\forall b_1, b_2, b_3 \in B$. Obviously,

$$(25) \quad f(b_1, b_2) = -f(b_2, b_1), \quad \forall b_1, b_2 \in B.$$

Since

$$\sigma(f(b_1, b_2)) = 0, \quad \sigma(g(b_1, b_2, b_3)) = 0, \quad \forall b_1, b_2, b_3 \in B,$$

it results that $f(b_1, b_2), g(b_1, b_2, b_3) \in \rho(A)$. So, the following two multilinear functions may be defined by

$$(26) \quad f_1 = \rho^{-1} \circ f, \quad g_1 = \rho^{-1} \circ g.$$

Because E is a BT -algebra, (f_1, g_1) will satisfy the following relations:

$$(27) \quad f_1(b_1, b_2) = -f_1(b_2, b_1), \quad \forall b_1, b_2 \in B;$$

$$(28) \quad \begin{aligned} &g_1(b_1, b_2, b_3) + g_1(b_2, b_3, b_1) + g_1(b_3, b_1, b_2) - g_1(b_1, b_3, b_2) - \\ &- g_1(b_3, b_2, b_1) - g_1(b_2, b_1, b_3) = f_1(b_1, [b_2, b_3]) + f_1(b_2, [b_3, b_1]) + \\ &+ f_1(b_3, [b_1, b_2]) + \phi(b_1)(f_1(b_2, b_3)) + \phi(b_2)(f_1(b_3, b_1)) + \\ &+ \phi(b_3)(f_1(b_1, b_2)), \quad \forall b_1, b_2, b_3 \in B. \end{aligned}$$

Definition 15. The pair of functions (f_1, g_1) , $f_1: B \times B \rightarrow A$, $g_1: B \times B \times B \rightarrow A$ satisfying (27) and (28) is called — according to Eilenberg [5] — the *factor set* of the extension $(\mathcal{E})$ corresponding to the section θ .

In connection with the preceding definition, it must be remarked that the representation given by formulas (22.1 — 7) may be changed by any $R \in \mathfrak{R}(B, A)$. This leads us to introduce

Definition 16. Let $R \in \mathfrak{R}(B, A)$ and the mappings $f_1: B \times B \rightarrow A$, $g_1: B \times B \times B \rightarrow A$ satisfying (27) and (28), with the first component of R as ϕ in (28). Then the pair (f_1, g_1) will be called the *factor set corresponding to R* .

Let us denote $M_R(B, A) = \underset{\text{not}}{M_R}$ the set of all the factor sets corresponding to R .

If θ is changed by the section $\tilde{\theta}$ then there exists the linear mapping $h: B \rightarrow E$ such that $\tilde{\theta} = \theta + h$. Because $h(b) \in \rho(A)$, $\forall b \in B$, it is possible and useful to define the function $h_1: B \rightarrow A$ by

$$(29) \quad h_1 = \rho^{-1} \circ h.$$

Let us now consider $(\tilde{f}_1, \tilde{g}_1)$ the factor set corresponding to $\tilde{\theta}$. Then

$$(30) \quad \tilde{f}_1(b_1, b_2) - f_1(b_1, b_2) = \phi(b_1)(h_1(b_2)) - \phi(b_2)(h_1(b_1)) - h([b_1, b_2]);$$

$$(31) \quad \begin{aligned} &\tilde{g}_1(b_1, b_2, b_3) - g_1(b_1, b_2, b_3) = \phi_1(b_1)(h_1(b_2), h_1(b_3)) + \\ &+ \phi_2(b_2)(h_1(b_1), h_1(b_3)) + \phi_3(b_3)(h_1(b_1), h_1(b_2)) + \\ &+ \phi_{12}(b_1, b_2)(h_1(b_3)) + \phi_{13}(b_1, b_3)(h_1(b_2)) + \phi_{23}(b_2, b_3)(h_1(b_1)) - \\ &- h_1(<b_1, b_2, b_3>), \quad \forall b_1, b_2, b_3 \in B. \end{aligned}$$

By using the notations

$$(32) \quad (\delta h_1)(b_1, b_2) = \phi(b_1)(h_1(b_2)) - \phi(b_2)(h_1(b_1)) - h_1([b_1, b_2]);$$

$$(33) \quad \begin{aligned} (\delta' h_1)(b_1, b_2, b_3) = & \phi_1(b_1)(h_1(b_2), h_1(b_3)) + \phi_2(b_2)(h_1(b_1), h_1(b_3)) + \\ & + \phi_3(b_3)(h_1(b_1), h_1(b_2)) + \phi_{12}(b_1, b_2)(h_1(b_3)) + \phi_{13}(b_1, b_3)(h_1(b_2)) + \\ & + \phi_{23}(b_2, b_3)(h_1(b_1)) - h_1(\langle b_1, b_2, b_3 \rangle), \quad \forall b_1, b_2, b_3 \in B. \end{aligned}$$

Eqs. (30) and (31) become

$$(34) \quad \tilde{f}_1 - f_1 = \delta h_1, \quad \tilde{g}_1 - g_1 = \delta' h_1.$$

Definition 17. The pair of functions $(\delta h_1, \delta' h_1)$, $\delta h_1: B \times B \rightarrow A$, $\delta' h_1: B \times B \times B \rightarrow A$, just introduced by (32) and (33), is called a *coboundary* of h_1 associated to the representation corresponding to an Abelian extension. If we change in (33) ϕ by the first component of $R \in \mathfrak{R}(B, A)$ we obtain the *coboundary* of h_1 associated to R .

Definition 18. The factor sets (f_1, g_1) , $(\tilde{f}_1, \tilde{g}_1)$ of M_R are *equivalent* if they differ by a coboundary.

Proposition 16. The binary relation defined on M_R by Definition 18 is an *equivalence relation*.

If we denote by δ the binary relation on M_R above defined, then $M_{R/\delta}$ will denote the quotient set of M_R relative to δ .

It follows from (34) the

Proposition 17. The factor sets associated to a modularly-split Abelian extension are *equivalent with each other*.

Definition 19. The equivalence class of the factor set (f_1, g_1) is called the *factor class* of the extension $(\mathcal{E})$.

Proposition 18. The modularly-split Abelian extensions of the BT-algebra B by the K -module A inducing the same representation of B on A are *equivalent if and only if the factor classes of both extensions are coincident*.

Corollary 1. The correspondence between the equivalence classes of modularly-split Abelian extensions of BT-algebra B by K -module A inducing the same representation of B on A and the factor classes is *injective*.

Proposition 19. Let A be a K -module, B a BT-algebra over K , $R = (\phi, \phi_i, \phi_{ij})$, $(i, j = 1, 2, 3; i < j)$ a representation of B in A and (f_1, g_1) a factor set for R . Then there exists a modularly-split Abelian extension of B by A such that R is the representation induced by this extension.

The proof uses the following construction.

Let A be a K -module, B a BT-algebra, $R = (\phi, \phi_i, \phi_{ij})$, $(i, j = 1, 2, 3; i < j)$ a representation of B in A and (f_1, g_1) a factor set of representation R . Let us define

on $E = A \times B$ the multilinear operations

$$\begin{aligned} [(a_1, b_1), (a_2, b_2)] &= (\phi_{b_1}(a_2) - \phi_{b_2}(a_1) + f_1(b_1, b_2), [b_1, b_2]), \\ \langle (a_1, b_1), (a_2, b_2), (a_3, b_3) \rangle &= (\phi_{12}(b_1, b_2)(a_3) + \phi_{13}(b_1, b_3)(a_2) + \\ &+ \phi_{23}(b_2, b_3)(a_1) + \phi_1(b_1)(a_2, a_3) + \phi_2(b_2)(a_1, a_3) + \phi_3(b_3)(a_1, a_2) + \\ &+ g_1(b_1, b_2, b_3), \langle b_1, b_2, b_3 \rangle), \quad \forall (a_1, b_1), (a_2, b_2), (a_3, b_3) \in A \times B. \end{aligned}$$

From Propositions 14, 15, 17, 18 and 19 it follows

Theorem 2. *The set of all equivalence classes of modularly-split Abelian extensions of a BT-algebra B by the K -module A , inducing the same representation $R \in \mathcal{R}(B, A)$, are in 1-1 correspondence with the quotient set $M_{R/\delta}$.*

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ALGEBRE TRIPLE BINARE

(Rezumat)

Se studiază reprezentările și extensiile algebrelor triple binare.

GRADIENT MEASURE ON STANDARD H -CONE

BY

LILIANA POPA

Let S be a standard H -cone of functions on a semi-saturated set X . We shall use the notions and notations of [1]. In the most general case on the set X we don't have a differential structure by the mean of which to define a notion of gradient; therefore it can be used a correspondence between the elements of the H -cone and a suitable set of measures, which is called *measure of representation*.

In [5] we have defined a notion of measure of representation on a standard autodual H -cone, denoted by σ . In the presence of the axiom D, we have proved the positivity of the gradient measure. We drop the hypothesis of autoduality and the axiom D; such we obtain in a most general situation the positivity of the gradient measure.

If we consider the canonical measure of representation which was constructed in [7], we get in the presence of the axiom D_0 the inequality

$$\int f d\sigma(f) \geq 0$$

for any f , a difference of bounded potentials.

Let S be a standard H -cone of functions on a set X , semi-saturated set X . We denote

$$S_M = \{s \in S \mid \exists \alpha \in \mathbb{R}, s \leq \alpha\}$$

and let $[S_M] = S_M - S_M$. Let R be the algebra of the locally bounded elements of S , where

$$R = \{f: X \rightarrow \mathbb{R} \mid \exists U_i, i \in I, \text{ naturally open such that } f|_{U_i} \in [S_M(U_i)]\}.$$

We suppose that the map

$$G \rightarrow S(G), \quad G \subseteq X, \text{ naturally open}$$

is a sheaf, that means the H -cone satisfies *natural sheaf property*. $S(G)$ was defined in [2].

Definition. The family of applications $\sigma = (\sigma_U)_{U \subseteq X, U \text{ naturally open}}$

$$\sigma_U: [S(U)] \rightarrow M(U),$$

where $M(U)$ is the set of all σ -finite measures on U , is called a measure of representation, if

1. σ_U is linear;
2. $\forall f, g \in S(U)$, with $f \leq g$, the next equivalence is valid

$$f \leq g \Leftrightarrow \sigma_U(f) \leq \sigma_U(g);$$
3. for any U_1, U_2 naturally open, with $U_1 \subset U_2$ and $f \in S(U_2)$

$$\sigma_{U_1}(f|_{U_1}) = \sigma_{U_2}(f)|_{U_1}.$$

We know from [6] that the measure of representation satisfies the following inequality

$$(1) \quad \sum_{k=0}^n (-1)^{k+1} C_n^k f^{n-k} \sigma(f^k) \geq 0.$$

This inequality was proved in an harmonic space by M a e d a [4] using the hypothesis that $\sigma(1)=0$.

As a consequence, the next equality takes place:

$$(2) \quad \begin{aligned} &\sigma(fgh) - f\sigma(gh) - g\sigma(fh) - h\sigma(fg) + fg\sigma(h) + \\ &+ fh\sigma(g) + gh\sigma(f) - fgh\sigma(1) = 0. \end{aligned}$$

If σ is a measure of representation of R and h is a function on X , $h > 0$, then the application

$$\sigma'(f) = h\sigma(f)$$

defines another measure of representation.

L e m m a 1. Let $f, g, h \in R$, $h > 0$, then the measure

$$\frac{1}{2h} (f\sigma(gh)g\sigma(fh) - \sigma(fgh) - fg\sigma(h))$$

does not depends on h .

P r o o f. Because $1 \in R$, from (2) we deduce

$$\frac{1}{2h} (f\sigma(gh) + g\sigma(fh) - \sigma(fgh) - fg\sigma(h)) = \frac{1}{2} (f\sigma(g) + g\sigma(f) - \sigma(fg) - fg\sigma(1)).$$

Therefore we can give the next

D e f i n i t i o n. For any $f, g \in R$, the measure

$$\delta_{[f,g]} = \frac{1}{2} (f\sigma(g) + g\sigma(f) - \sigma(fg) - fg\sigma(1))$$

is called *mutual gradient measure*.

The mapping $(f, g) \rightarrow \delta_{[f,g]}$ is symmetric and bilinear.

D e f i n i t i o n. The measure $\delta_f = \delta_{[f,g]}$ is called *gradient measure of f* and has the expression

$$\delta_f = \frac{1}{2} (f\sigma(f) - \sigma(f^2) - f^2\sigma(1)).$$

T h e o r e m 1. $\delta_f \geq 0$ on X , for any $f \in R$.

The assertion is a consequence of the inequality (1), for $n=2$. From the positivity of the gradient measure, we deduce the next

P r o p o s i t i o n 1.

a. If $f, g \in R$ and $\lambda > 0$, then $|\delta_{[f,g]}| \leq \frac{1}{2}(\lambda \delta_f + \lambda^{-1} \delta_g)$.

b. If $f, g \in R$ and $A \subset X$ is borelian, then

$$(|\delta_{[f,g]}|(A))^2 \leq \delta_f(A) \delta_g(A).$$

c. If $f \in R$ and $\delta_f = 0$ on X , then $\delta_{[f,g]} = 0$ for all $g \in R$.

For the proof, see [4].

Let $R^{(h)} = \{f/h \mid f \in R\}$ and $\sigma^{(h)}(f) = \sigma(fh)$ the associated measure of representation.

Proposition 2. Let $h \in R$, $h > 0$ on X , then

$$\delta_{[f,g]}^{(h)} = h \delta_{[f,g]}$$

for any $f, g \in R^{(h)}$, where $\delta_{[f,g]}^{(h)}$ is the gradient measure relative to the measure of representation $\sigma^{(h)}(f)$.

Proof. Because $1 \in R^{(h)}$, we have

$$\begin{aligned} \delta_{[f,g]}^{(h)} &= \frac{1}{2} (f\sigma^{(h)}(g) + g\sigma^{(h)}(f) - \sigma^{(h)}(fg) - fg\sigma^{(h)}(1)) = \\ &= \frac{1}{2} (f\sigma(g h) + g\sigma(fh) - \sigma(fgh) - fg\sigma(h)) = h \delta_{[f,g]}. \end{aligned}$$

Theorem 2. For any $f, g, h \in R$, we have

$$\delta_{[f,g,h]} = f\delta_{[g,h]} + g\delta_{[f,h]}.$$

Proof. The proof is immediate, using the definition of the mutual gradient measure and the inequality (2). We have indeed

$$\begin{aligned} f g \sigma(h) + h \sigma(fg) - \sigma(fgh) - f g h \sigma(1) - f(g\sigma(h) + h\sigma(g) - \\ - \sigma(hg) - h g \sigma(1)) - g(f\sigma(h) + h\sigma(f) - \sigma(fh) - f h \sigma(1)) = 0. \end{aligned}$$

Let $S = S^{**}$ and let E be the canonical Green space constructed in [1]. We suppose:

I. Any pure potential of S is a Green potential. (Hence $S_P \subset P(E)$.)

II. S satisfies the axiom of polarity on E .

III. For any $U \subset E$, naturally open, $S(U)$ satisfies the axiom A. These hypothesis were supposed in [7], where we have constructed a canonical measure of representation.

If $U \subset E$ is naturally open, for any $f \in S(U)$, $f = h + p$, with $h \in S_H(U)$, $p \in S_P(U)$. For any $p \in S_P(U)$, we denote by $\sigma_U(p)$ the H -measure on U , for which

$$p(x) = \int G^U(x, y) d\sigma_U(p)(y),$$

where $G^U(x, y)$ is the canonical Green function [7]. We have defined

$$\sigma_U(f) = \sigma_U(p).$$

We generalize some assertions of [7].

Proposition 3. *If (S, X) satisfies the axiom D_0 , then σ_U satisfies the axiom 3 of the definition of a measure of representation.*

The proof follows the same arguments of [7], Theorem 8, where the axiom D can be replaced by the axiom D_0 .

Proposition 4. *If σ is the canonical measure of representation, then*

$$\int f d\sigma(f) \geq 0,$$

for any f which is a difference of two bounded potentials.

Proof. Because the gradient measure is positive, that is

$$f\sigma(f) - f^2\sigma(1) - \sigma(f^2) \geq 0,$$

we have

$$\int f d\sigma(f) \geq \int f^2 d\sigma(1) + \sigma(f^2)(1) \geq 0.$$

The last inequality follows from the fact that f^2 is a difference of bounded potentials and from [7], we know that if $p' \leq q'$, it results $\sigma(p')(1) \leq \sigma(q')(1)$.

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MĂSURI GRADIENT PE UN H-CON STANDARD

(Rezumat)

Se definește noțiunea de măsură gradient în cadrul mai general al unui H -con standard de funcții pe o mulțime semi-saturată, care satisface axioma de fascicol natural. Pozitivitatea se păstrează renunțând la axioma D și la ipoteza că $\sigma(1)=0$.

2-II-STRUCTURES SUR DES ESPACES À MÉTRIQUES RELATIVISTES

BY

ELENA PUȘCAȘU

1. Introduction. Soit

$$(1) \quad R_{\alpha\beta} = \frac{1}{2} R g_{\alpha\beta} + \chi T_{\alpha\beta}, \quad (\alpha, \beta, \dots = 0, 1, 2, 3)$$

les équations gravitationnelles d'Einstein de la théorie de la relativité, où: $g_{\alpha\beta}$ est le tenseur métrique d'un espace Riemann avec 4 dimensions, V_4 ; $R_{\alpha\beta}$ est le tenseur Ricci de cet espace; R — l'invariant Ricci; $T_{\alpha\beta}$ — le tenseur énergie-impulsion et χ une constante universelle (positive).

En supposant le tenseur $T_{\alpha\beta}$ connu, (1) représente un système d'équations avec dérivées partielles de deuxième ordre aux inconnues $g_{\alpha\beta}$ et, le principal problème de la théorie de la relativité est de déterminer le tenseur $g_{\alpha\beta}$. Une fois déterminé $g_{\alpha\beta}$, est connue la métrique ds^2 de l'espace V_4 . Le système (1) est, en général, difficile à résoudre et on connaît les solutions de ce système dans des cas spéciaux [1], [4]...[6].

Dans cette Note on donne une solution des équations d'Einstein et on étudie quelques propriétés géométriques de cette solution.

2. Une solution des équations d'Einstein. Nous supposons la métrique de l'espace V_4 statique et avec une symétrie sphérique. Dans des coordonnées polaires elle a la forme

$$(2) \quad ds^2 = e^{2n} c^2 dt^2 - e^{2l} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2),$$

où n et l sont de fonctions seulement de r [4].

Pour la métrique (2) les équations (1) ont la forme

$$e^{2n-2l} \left[\frac{2l'}{r} - \frac{1}{r^2} \right] + \frac{1}{r^2} e^{2n} = \chi T_{00},$$

$$(3) \quad \frac{2n'}{r} + \frac{1}{r^2} - \frac{1}{r^2} e^{2l} = \chi T_{11},$$

$$r^2 e^{-2l} \left[n'' + n'/2 - l'/n + \frac{n'-l'}{r} \right] = \chi T_{22}.$$

Le tenseur énergie-impulsion $T_{\alpha\beta}$ englobe un terme d'origine électromagnétique

représenté par le tenseur de Maxwell $\tau_{\alpha\beta}$, et un terme qui donne la distribution de masse représenté par le tenseur matériel $M_{\alpha\beta}$ [1].

Nous avons

$$(4) \quad T_{\alpha\beta} = \tau_{\alpha\beta} + M_{\alpha\beta}.$$

Pour la métrique (2) le tenseur de Maxwell possède les composantes:

$$\tau_{00} = \frac{h^2}{2r^4} e^{-2l}, \quad \tau_{11} = \frac{h^2}{2r^4} e^{-2n},$$

(5)

$$\tau_{22} = \frac{h^2}{2r^2} e^{2n-2l}, \quad \tau_{33} = \tau_{22} \sin^2 \theta,$$

où h est une constante qui caractérise la particule qui crée le champ électromagnétique [4].

En suite, nous supposons que le tenseur $M_{\alpha\beta}$ détermine sur V_4 une 2-II-structure, c'est-à-dire, nous avons

$$(6) \quad M_{\alpha}^{\gamma} M_{\gamma}^{\beta} = \lambda^2 \delta_{\alpha}^{\beta} \quad (\lambda \text{ étant une constante arbitraire}).$$

De (4)...(6) il en résulte

$$T_{00} = \frac{h^2}{2r^4} e^{-2l} + |\lambda| e^{2n}, \quad T_{11} = -\frac{h^2}{2r^4} e^{-2n} - |\lambda| e^{2l},$$

(7)

$$T_{22} = \frac{h^2}{2r^2} e^{-2n-2l} - |\lambda| r^2, \quad T_{33} = T_{22} \sin^2 \theta.$$

Pour $T_{\alpha\beta}$ donné par (7), de (3) on obtient

$$(8) \quad e^{2n} = 1 + \frac{a}{r} + \frac{\chi h^2}{2r^2} - \frac{\chi |\lambda| r^2}{3}, \quad e^{2l} = e^{-2n},$$

où a est une constante d'intégration.

Donc, l'espace V_4 a la métrique

$$(9) \quad ds^2 = \left[1 + \frac{a}{r} + \frac{\chi h^2}{2r^2} - \frac{\chi |\lambda| r^2}{3} \right] c^2 dt^2 - \left[1 + \frac{a}{r} + \frac{\chi h^2}{2r^2} - \frac{\chi |\lambda| r^2}{3} \right]^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

Remarques. 1. Pour $h=\lambda=0$ on obtient $e^{2n}=1+a/r$, donc l'espace de Schwarzschild.

2. Pour $\lambda=0$ on obtient $e^{2n}=1+a/r+\chi h^2/2r^2$, donc l'espace de Reissner-Weyl.

3. Pour $h=0$ on obtient $e^{2n}=1+a/r-\chi|\lambda|r^2/3$, donc l'espace d'Einstein.

4. Pour $a=h=0$ on obtient $e^{2n}=1-\chi|\lambda|r^2/3$, donc l'espace de Sitter.

3. Propriétés géométriques de la solution (9).

P_1 . L'invariant Ricci de la métrique (9) est constant. Vraiment, de (9) on obtient

$$(10) \quad R = -4\chi|\lambda| = \text{const.}$$

P₂. La distribution donnée par le tenseur $T_{\alpha\beta}$ est normale.

Les valeurs propres du tenseur $T_{\alpha\beta}$ sont données par

$$(11) \quad s_0 = s_1 = \frac{h^2}{2r^4} + |\lambda|, \quad s_2 = s_3 = -\frac{h^2}{2r^4} + |\lambda|$$

et les vecteurs propres correspondants sont données par

$$(12) \quad \begin{aligned} u(u_\alpha), \quad u_\alpha &= (\sqrt{2} e^n, e^l, 0, 0); \\ v_1(v_{1\alpha}), \quad v_{1\alpha} &= (e^n, \sqrt{2} e^l, 0, 0); \\ v_2(v_{2\alpha}), \quad v_{2\alpha} &= (0, 0, r \sin \theta, -r \sin \theta \cos \theta); \\ v_3(v_{3\alpha}), \quad v_{3\alpha} &= (0, 0, -r \cos \theta, -r \sin^2 \theta). \end{aligned}$$

On montre que les vecteurs (12) sont perpendiculaires et

$$(13) \quad \|u\|^2 = 1, \quad \|v_1\|^2 = \|v_2\|^2 = \|v_3\|^2 = -1.$$

Donc, le vecteur u est orienté en temps et les vecteurs v_1, v_2, v_3 sont orientés en espace.

Parce que le tenseur $T_{\alpha\beta}$ a un vecteur propre orienté en temps la distribution donnée par $T_{\alpha\beta}$ est normale.

P₃. L'espace physique associé à l'espace V_4 admet un groupe de mouvements G à trois paramètres.

Vraiment, l'espace physique F au moment t , associé à l'espace V_4 , a la métrique

$$(14) \quad d\sigma^2 = \left[1 + \frac{a}{r} + \frac{\chi h^2}{2r^2} - \frac{\chi |\lambda| r^2}{3} \right]^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

On montre que l'espace F admet trois vecteurs Killing linéaires indépendants ξ_1, ξ_2, ξ_3 qui sont donnés par

$$(15) \quad \begin{aligned} \xi_1 &= (0, \cos \varphi, -\sin \varphi \operatorname{ctg} \theta), \\ \xi_2 &= (0, \sin \varphi, \cos \varphi \operatorname{ctg} \theta), \\ \xi_3 &= (0, 0, 1), \end{aligned}$$

donc, F admet un groupe de mouvements G à trois paramètres.

P₄. Le groupe G est parfait.

Si nous notons par X_1, X_2, X_3 les opérateurs infinitésimaux du groupe G , les équations de structure du groupe G sont données par

$$(16) \quad (X_1, X_2) = -X_3, \quad (X_1, X_3) = X_2, \quad (X_2, X_3) = -X_1,$$

d'où résulte que G est un groupe parfait.

P₅. Les géodésiques de l'espace F sont données par

$$\begin{aligned}
 & \frac{d^2 r}{d\sigma^2} - n' \left[\frac{dr}{d\sigma} \right]^2 - re^{2n} \left[\frac{d\theta}{d\sigma} \right]^2 - re^{2n} \sin^2 \theta \left[\frac{d\varphi}{d\sigma} \right]^2 = 0, \\
 (17) \quad & \frac{d^2 \theta}{d\sigma^2} + \frac{2}{r} \frac{dr}{d\sigma} \frac{d\theta}{d\sigma} - \sin \theta \cos \theta \left[\frac{d\varphi}{d\sigma} \right]^2 = 0, \\
 & \frac{d^2 \varphi}{d\sigma^2} + \frac{2}{r} \frac{dr}{d\sigma} \frac{d\varphi}{d\sigma} + 2 \operatorname{ctg} \theta \frac{d\theta}{d\sigma} \frac{d\varphi}{d\sigma} = 0.
 \end{aligned}$$

On remarque que les courbes $r(\theta=\varphi=\text{const.})$ sont des géodésiques et elles coupent orthogonalement les hypersurfaces $r=\text{const.}$

P_6 . La métrique (9) est plongée dans un espace pseudoeuclidien

$$(18) \quad d\rho^2 = dz_1^2 + dz_2^2 - dz_3^2 - dz_4^2 - dz_5^2 - dz_6^2,$$

où

$$\begin{aligned}
 (19) \quad & z_1 = \sqrt{1+\tau} \sin ct, \quad z_2 = \sqrt{1+\tau} \cos ct, \quad z_3 = f(r), \\
 & z_4 = r \cos \varphi \sin \theta, \quad z_5 = r \sin \varphi \sin \theta, \quad z_6 = r \cos \theta,
 \end{aligned}$$

avec [3]

$$(20) \quad \tau = \frac{a}{r} + \frac{\chi h^2}{2r^2} - \frac{\chi |\lambda| r^2}{3}, \quad f'(r) = \frac{\tau'^2 - 4\tau}{4(1+\tau)}.$$

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2-II-STRUCTURI PE SPAȚII CU METRICI RELATIVISTE

(Rezumat)

În prima parte a lucrării se determină o soluție V_4 a ecuațiilor gravitaționale ale lui Einstein, în ipoteza că tensorul material $M_{\alpha\beta}$ determină pe V_4 o 2-II-structură. În ultima parte a lucrării se studiază o serie de proprietăți geometrice ale acestei soluții.

APPLICATIONS OF THE NON-STANDARD ANALYSIS

BY

PETRU DRAGOȘ

0. Introduction. Starting from the field $\mathbb{R}$ of the real number as a model for the formal system of the theory of the real ordered closed fields, complete in the topological sense and Archimedean, A. Robinson [15] has found the extension ${}^*\mathbb{R}$ which is *ordered, nonarchimedean, not complete* and, moreover, the models of ${}^*\mathbb{R}$ are *not isomorphic*. We work on a model [6] based on an δ -incomplete ultrafilter, fixed and built as a consequence of the Lowenheim-Skolem theorem [5] which states that any infinite structure has an elementary extension. We use the three principles: of transfer, of idealization and of standardization, and the rules stated in [13].

Using relevant examples, we point out the efficiency of the infinitesimal methods, resulted from the non-standard analysis in the study of geometry. The fundamental object is the point, which is supposed to be non-standard and finite [9]. The point defines a standard frame and we consider that through the point is passing a differentiable manifold. The exposure refers to subtle notions, but taking benefit of the non-standard method they look to be very simple.

1. Differentiable Curves. **Definition 1.** Let $({}^*\varepsilon_2, {}^*\mathbf{E}, {}^*\varphi)$ be the non-standard Euclidean plane $\mathfrak{R} = (O; B = (\mathbf{i}, \mathbf{j}))$ an orthonormal frame and ${}^*h_{\mathfrak{R}}: {}^*\varepsilon_2 \rightarrow {}^*\mathbb{R}^2$ the standard system of Cartesian coordinates defined by $\mathfrak{R}$. Let ${}^*I \subset {}^*\mathbb{R}$ be a closed interval and $c: {}^*I \rightarrow {}^*\varepsilon_2$ an application. We consider the linear space ${}^*\mathbb{R}^2$ and the standard real, vector valued function $h_{\mathfrak{R}} \circ c = \mathbf{r}: {}^*I \rightarrow {}^*\mathbb{R}^2$. For any $t \in {}^*I$ we have $\mathbf{r}(t) = (x(t), y(t)) \in {}^*\mathbb{R}^2$, where $(x(t), y(t))$ are the coordinates of the point $c(t) \in \gamma = c({}^*I)$. There consequently appear two standard real functions of real variable, written $x, y: {}^*I \rightarrow \mathbb{R}$ and defined by

$$(1) \quad \begin{aligned} t &\rightarrow x(t), \\ t &\rightarrow y(t), \end{aligned} \quad \forall t \in {}^*I.$$

We consider on ${}^*\mathbb{R}$ the topology of the monads [12] and on *I the induced topology. The vector function $\mathbf{r}$ is called of class C^k on *I , $k=0, 1, \dots, \infty$ if its component functions x, y are of class C^k .

Definition 2. Any application $c: {}^*I \rightarrow {}^*\varepsilon_2$, with $\mathbf{r} = h_{\mathfrak{R}} \circ c: {}^*I \rightarrow {}^*\mathbb{R}^2$ of

$(\text{arc } MM') = 1/R = c = \text{constant}$, independently on the fact that M, M' are or not in the same monad.

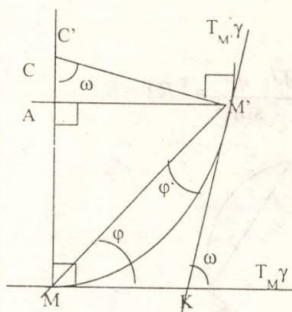


Fig. 2

Definition 4. Let γ be a planar curve, M be an ordinary point of γ and $\mu(M)$ be the monad of M . The circle with the properties (1) contains the points and have the same tangent at M ; it is called the *circle of curvature* at M .

Proposition 3. The circle of curvature in an ordinary point $M \in \gamma$ of the curve γ has the radius $R = \lim_{M' \rightarrow M} \text{length}(\text{arc } MM')/\omega$, and its center C

is situated on the normal in M at γ , s.t. $\|MC\| = R$.

Proof. Let C, C' be the center of curvature for M and M' from the same monad and $M'A \perp MC$ (Fig. 2).

In the triangle $\Delta AMM' \Rightarrow |AM'| \equiv |MM'| \sin \angle AMM'$.

In the triangle $\Delta C'AM' \Rightarrow |AM'| \equiv |C'M'| \sin \angle MC'M'$.

Because the quadrilateral $MKM'C'$ has its vertices on the circle there results
 $\text{ang } MC'M' = \omega \Rightarrow |MM'| \cos \varphi \equiv |C'M'| \sin \omega \Rightarrow |C'M'| \equiv \frac{|MM'| \cos \varphi}{\sin \omega} \equiv$

$$\equiv \frac{|MM'|}{\text{arc } MM'} \cdot \frac{\text{arc } MM'}{\omega} \cdot \frac{\omega}{\sin \omega} \cos \varphi \Rightarrow \lim_{M' \rightarrow M} \|C'M'\| = \lim_{M' \rightarrow M} \frac{\text{length}(\text{arc } MM')}{\omega} = R.$$

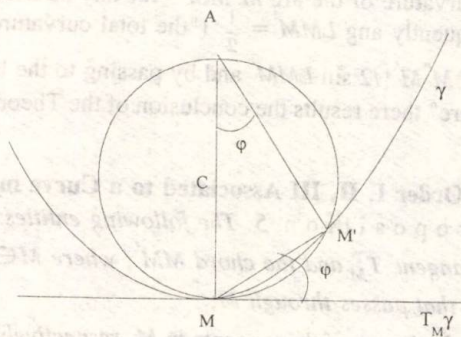


Fig. 3

Remark 4. The limit of the intersection point between the normal at an ordinary point and the normal of γ at an ordinary point $M \in \gamma$ and the normal of γ at $M' \in \gamma \cap \mu(M)$, when $M' \rightarrow M$, is the center of the circle of curvature of γ at M .

3. The Osculating Circle.

Let γ be a curve, $M \in \gamma$ an ordinary point of the curve and T_M^γ the tangent at the curve in the point M . We consider the circle

$C(M, M', T_M^\gamma)$ that passes through $M \in \gamma$ and $M' \in \gamma \cap \mu(M)$ it is tangent at T_M^γ (Fig. 3).

Proposition 4. The circle $C(M, M', T_M^\gamma)$ tends to the "circle of the

curvature" for $M' \rightarrow M$.

P r o o f. There are true the relations

$$\begin{aligned} |MM'| &\equiv |AM| \sin \varphi \equiv 2 |MC| \sin \varphi \Rightarrow |MC| \equiv \\ &\equiv \frac{|MM'|}{2 \sin \varphi} \Rightarrow \lim_{M' \rightarrow M} |MC| = \lim_{M' \rightarrow M} \frac{|MM'|}{\text{arc } MM'} \frac{\text{arc } MM'}{2 \varphi} \frac{\varphi}{\sin \varphi} = 1 R 1 = R. \end{aligned}$$

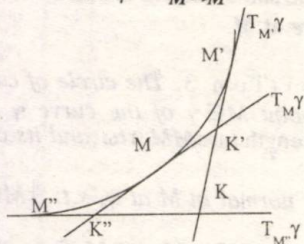


Fig. 4

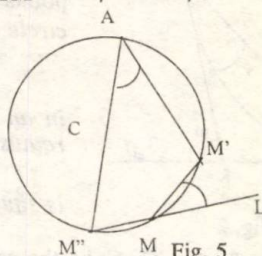


Fig. 5

We consider now the tangent circle in M at the tangent T_M^γ of the curve γ (M is an ordinary point) and containing M' , $M'' \in (\gamma) \cap \mu(M)$: M' , M'' are situated on γ on one side and the other of M . This circle is uniquely determined by the three points M , M' , M'' and is called the *osculating circle* of the curve γ in the point M .

T h e o r e m 1. *The osculating circle at a curve in an ordinary point is the standard part of the "circles of curvature".*

P r o o f. There is true the relation (Fig. 4)

$$\text{ang } T_M // K K' = \text{ang } T_M // K' K' + \text{ang } K // K' K'.$$

From $\text{ang } M'' M K'' + \text{ang } K' M M' = \pi - \text{ang } M'' M M'$ it results that $\text{ang } M'' M M'$ is the supplement of half of the "total curvature of the arc $M'' M M'$ " for any M situated between M' and M'' (Fig. 5). Consequently $\text{ang } L M M' = \frac{1}{2}$ ("the total curvature of the arc $M'' M M'$ "). By using $|M'' C| \equiv |M'' M'|/2 \sin L M M'$ and by passing to the limit in the ratio $M'' M' / \frac{1}{2}$ "the total curvature" there results the conclusion of the Theorem.

4. Infinitesimal Elements of Order I, II, III Associated to a Curve in the Monad of a Point of the Curve. Proposition 5. *The following entities:*

i) *the angle φ formed by the tangent T_M^γ and the chord MM' , where $M \in \gamma$ is an ordinary point and $M' \in \gamma \cap \mu(M)$ that passes through M ;*

ii) $\|MK\|$, $\|M'K\|$ *are called the length of the tangents to M , respectively M' ;*

iii) $\|MK\| + \|M'K\|$;

iv) *the projection of the chord $|MM'|$ on a direction nonperpendicular to the tangent at M of the curve M , $M' \in \gamma \cap \mu(M)$ are infinitesimal elements of order I;*

P r o o f. i) There are true the implications (Fig. 6):

$$\begin{aligned}
 2 |MC| \sin \varphi &\equiv |MM'| \Rightarrow \frac{\sin \varphi}{|MM'|} = \frac{1}{2 |MC|} \Rightarrow \frac{\varphi}{\text{length}(\text{arc } MM')} = \\
 &= \frac{\varphi}{\sin \varphi} \frac{\sin \varphi}{|MM'|} \frac{|MM'|}{\text{length}(\text{arc } MM')} \Rightarrow \lim_{M' \rightarrow M} \frac{\varphi}{\text{length}(\text{arc } MM')} = \\
 &= \lim_{M' \rightarrow M} \frac{\varphi}{\sin \varphi} \frac{1}{2 |MC|} \frac{|MM'|}{\text{length}(\text{arc } MM')} = 1 \frac{1}{2R} 1 = \frac{1}{2R}.
 \end{aligned}$$

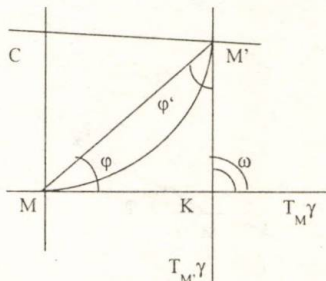


Fig. 6

The other statements are analogously proved.

Proposition 6. The following entities:

i) the distance from M' to $P'' \in T_M$ written $\|M'P''\| \sim s^2/2R$; here M is an ordinary point and s is the length of the arc;

ii) $\|MK\| - \|M'K\|$ (with the previous notations);

iii) if γ, γ' are curves having the same tangent T_M^γ at the common ordinary point M , then the distance between two points situated on different curves in the same monad of M , are infinitesimal elements of order II.

Proof. There are satisfied:

$$\begin{aligned}
 \text{i)} \quad \lim_{M' \rightarrow M} \frac{|M'P'|}{|MM'|^2} &= \lim_{M' \rightarrow M} \frac{|MM'| \sin P'MM'}{|MM'|^2} = \lim_{M' \rightarrow M} \frac{\sin P'MM'}{\|MM'\|} = \\
 &= \lim_{M' \rightarrow M} \frac{\sin P'MM'}{\text{ang } P'MM'} \frac{\text{ang } P'MM'}{|MM'|} = \frac{1}{2R};
 \end{aligned}$$

$$\begin{aligned}
 \frac{|M'P''|}{\sin M'MP''} &= \frac{|MM'|}{\sin M'P''M} \Rightarrow \frac{|M'P''|}{|MM'|} = \frac{\sin P''MM'}{\sin M'P''M} \Rightarrow \\
 &\Rightarrow \frac{\|M'P''\|}{\|MM'\|^2} = \frac{\sin P''MM'}{\text{ang } P''MM'} \frac{1}{\|MM'\|} \frac{1}{\sin M'P''M}
 \end{aligned}$$

and by passing to the limit there results

$$\|M'P''\| \sim \frac{(\text{length}(\text{arc } MM'))^2}{2R} \Rightarrow \|M'P''\| \sim \frac{s^2}{2R}.$$

The other statements are analogously proved.

Proposition 7. The following entities (Fig. 7):

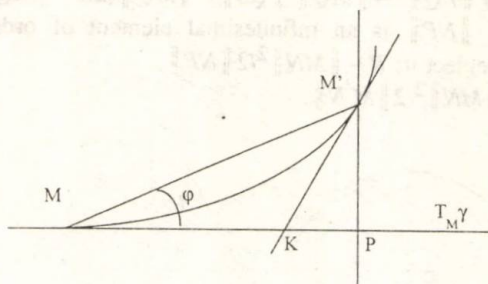


Fig. 10

ii) If $\|M'P\| \sim \lambda \|MP\|^{n+1}$, it is said that the curve γ and the tangent have a contact of order n and we obtain by computations that $\omega \sim (n+1)\text{ang } PMM'$. In this case $\|MK\|$, $\|KM'\|$ are not of the same order, nor by passing to the limit (like in the case that M is ordinary), and $\|M'P\|$ is an infinitesimal element of order $n+1$.

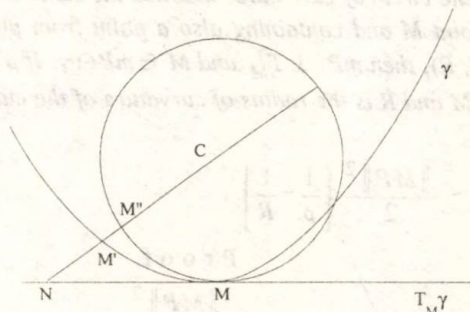


Fig. 11

ordinary point M of γ (Fig. 11) and $N \in \mu(M) \cap T_M \gamma$, $NC \cap (\gamma) = C'$. Then $R \sim \|MN\|^2 / 2 \|NM'\|$.

Proof. There are satisfied: $\|MN\|^2 = \|NM''\| (\|NM''\| + 2R)$; $\|NM''\|$ — is an infinitesimal element of order II; $\|NM'\|^2$ — is an infinitesimal element of order IV and we neglect it.

$$\Rightarrow \|MN\|^2 \sim \|NM\| R \Rightarrow R \sim \|MN\|^2 / 2 \|NM''\|, \text{ because } \|NM''\| \sim \|NM'\| \Rightarrow R \sim \|MN\|^2 / 2 \|NM'\|.$$

Proposition 9. Let $P \in \mu(M) \cap C(C)$, $C(C)$ — osculating circle with $PN \perp T_M \gamma$ and $M' \in PM \cap \gamma$ (Fig. 12).

Remark 5. i) We consider the case when the curve γ and the tangent $T_M \gamma$ have a contact of order

I; then

$$\begin{aligned} \frac{|M'P|}{|MP|^2} &= \\ &= \frac{|MM'| \sin \varphi}{|MM'|^2 \cos^2 \varphi} \sim \\ &\sim \frac{\sin \varphi}{|MM'|} = \lambda \neq 0 \Rightarrow \end{aligned}$$

$$\Rightarrow \|M'P\| \sim \lambda \|MP\|^2.$$

We remark that in the monad of μ if $M' \rightarrow M$ the angle φ is naturally stationary and $\varphi' \sim n\varphi$ and if $M \rightarrow M'$ the roles of φ and φ' are interchanged.

6. The Relation between the Circle of Curvature and the Osculating Circle. Applications. **Proposition 8.**

Let $C(C, R)$ be the osculating circle of the curve γ at and ordinary

Then the relation $R \sim \|MN\|^2/2 \|M'N\|$ holds.

Proof. We have (Fig. 12) $\|PQ\|^2 = \|MQ\| \|QS\| = \|MQ\| (2R - \|MQ\|) \Rightarrow \|NM\|^2 = \|NP\| (2R - \|NP\|)$. $\|NP\|$ is an infinitesimal element of order II $\Rightarrow \|NP\|^2$ is of order IV and we neglect it; $R \sim \|MN\|^2/2 \|NP\|$.

As $\|NP\| \sim \|NM'\| \Rightarrow R \sim \|MN\|^2/2 \|M'N\|$.

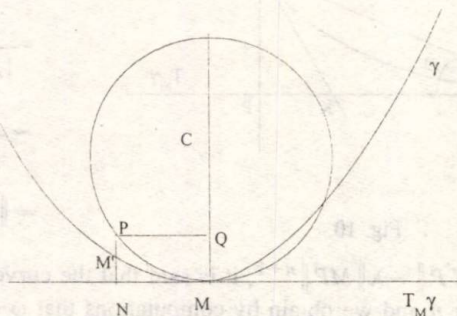


Fig. 12

Proposition 10. Let the circle of curvature, that has the same tangent with the curve γ at the ordinary point M and containing also a point from $\mu(M) \cap \cap(\gamma)$ (Fig. 13). If $m \in \mu(M) \cap C(C, R)$, then $mP \perp T_M \gamma$ and $M' \in mP \cap \gamma$. If ρ is the radius of curvature of the curve in M and R is the radius of curvature of the curve at M , then

$$\|mM'\| \sim \frac{\|MP\|^2}{2} \left(\frac{1}{\rho} - \frac{1}{R} \right).$$

Proof.

$$\rho \sim \frac{\|MP\|^2}{2 \|mP\|},$$

$$R \sim \frac{\|MP\|^2}{2 \|M'P\|} \Rightarrow \|mM'\| \sim \frac{\|MP\|^2}{2} \left(\frac{1}{\rho} - \frac{1}{R} \right).$$

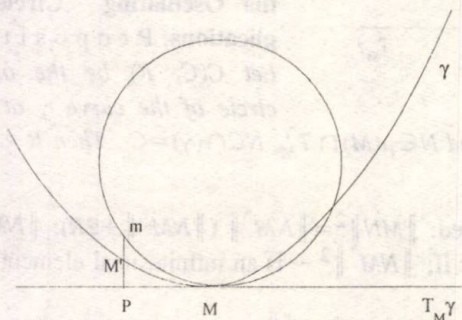


Fig. 13

Remark 6. a) For $\rho \neq R$, that is for the case in which the circle of curvature is not an osculating circle (satisfying weaker conditions), there results that $\|mM'\|$ is an infinitesimal element of order II.

b) For $\rho = R$, that is the circle of curvature is an osculating circle, then the

distance $\|mM'\|$ is an infinitesimal element of order at least III.

Corollary. Between all the circles that are tangent to the curve γ at M , the osculating circle at $M \in \gamma$ is the closest to the curve γ , because such a contact determines uniquely the circle. Because $\left(\frac{1}{\rho} - \frac{1}{R}\right) \frac{\|MP\|^2}{2}$ changes the sign at M , the osculating circle crosses the curve at the point M .

Application. To be studied the behaviour of the curve (γ) , given by $r(\varphi) = 3 \sin \varphi \cos \varphi / (\cos^3 \varphi + \sin^3 \varphi)$ in the double points.

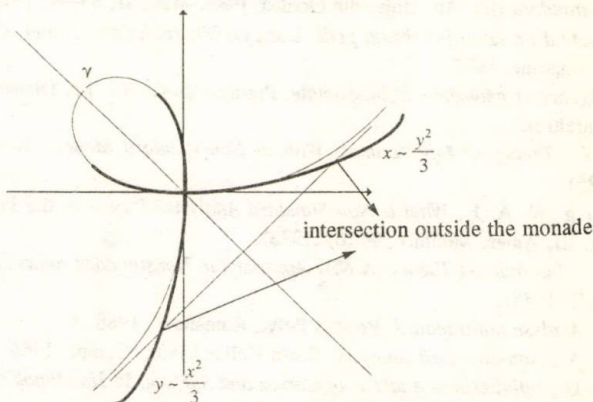


Fig. 14

Solution. O is the point of selfintersection (Fig. 14).

(1) $\varphi \rightarrow \pi/2 \Rightarrow \cos \varphi = \sin(\pi/2 + \varphi) \sin \varphi \sim 1 \Rightarrow \cos^3 \varphi + \sin^3 \varphi \sim 1$ so that $r(\varphi) \sim 3(\pi/2 + \varphi)$ and by passing to the limit in cartesian coordinates, we get $x \sim y^2/3$. For $\varphi \rightarrow -\pi/2$, the curve "lays" on the parabola $x \sim y^2/3$.

(2) $\varphi \rightarrow \pi$, the curve "lays" on the parabola $y \sim x^2/3$. We analogously prove for $\varphi \rightarrow 0$, that $y \sim x^2/3$, and for $\varphi \rightarrow \pi/2$, that $x \sim y^2/3$.

The efficiency of the non-standard method is obvious in the case of this application too.

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APLICAȚII ALE ANALIZEI NON-STANDARD

(Rezumat)

Folosind concepte noi, cum ar fi tangenta într-un punct standard la o curbă, cercul osculator sau elementul infinitezimal atașat unei curbe într-un punct standard al său, se studiază legăturile dintre elemente, subliniind simplitatea și eficiența calcului non-standard.

PROPERTIES CONCERNING A DIRECT METHOD FOR A JORDAN BASIS OF A LINEAR OPERATOR

BY

I. ONCIULESCU

0. Introduction. In this paper we shall denote by X_n a n -dimensional linear space over a commutative field K .

Let $f: X_n \rightarrow X_n$ be a linear operator. If $\lambda_0 \in K$ is an eigenvalue of the operator f , then a vector system

$$(1) \quad \{a_1, \dots, a_k\}$$

of X_n , satisfying

$$(1') \quad f(a_1) = \lambda_0 a_1, \quad f(a_2) = \lambda_0 a_2 + a_1, \dots, f(a_k) = \lambda_0 a_k + a_{k-1}, \quad a_1 \neq 0$$

will be called a series of eigenvectors and associated vectors.

A Jordan basis in X_n , corresponding to the operator f , is a basis in X_n of the form

$$(2) \quad \{a_1^1, a_2^1, \dots, a_{n_1}^1, \dots, a_1^p, a_2^p, \dots, a_{n_p}^p\}, \quad n_1 + n_2 + \dots + n_p = n,$$

where $\{a_1^i, a_2^i, \dots, a_{n_i}^i\}$, $i = \overline{1, p}$ is a series of eigenvectors and associated vectors, corresponding to the eigenvalue λ_i of the operator f . It is well known the following

Theorem (C. Jordan). *A Jordan basis in X_n , corresponding to the linear operator $f: X_n \rightarrow X_n$, exists if and only if the characteristic polynomial of f has all the roots in K [4], [5], [7].*

In [5] we have given an elementary proof of this theorem, analogous to the proof of [3].

In this paper we shall prove, for a such operator f , some properties related to the eigenvalues and the series of eigenvectors and associated vectors. These properties are useful in order to find, directly, some series of eigenvectors and associated vectors, with which a Jordan basis in X_n , of the operator f , can be determined.

1. The Number of Series of Eigenvectors and Associated Vectors and the Number of Vectors Corresponding to an Eigenvalue, in a Jordan Basis. Let $f: X_n \rightarrow X_n$ be a linear operator. Suppose that a Jordan basis in X_n , corresponding to f , exists and let it be of the form (2). Then we shall prove

Theorem 1. (A₁) *The operator f has not other eigenvalues than $\lambda_1, \dots, \lambda_p$.*

(A₂) *If the basis (2) is ordered so that*

$$(3) \quad \lambda_1 = \lambda_2 = \dots = \lambda_{m_1} \neq \lambda_{m_1+1}, \dots, \lambda_p$$

(hence m_1 is the number of the series corresponding to the eigenvalue λ_1) and the multiplicity of λ_1 , as root of the characteristic polynomial of f , is q_1 , then

$$(4) \quad m_1 = \dim S_{\lambda_1}(f),$$

where $S_{\lambda_1}(f)$ is the proper subspace of f , corresponding to λ_1 ,

$$(5) \quad S_{\lambda_1}(f) = \{x, x \in X_n, f(x) = \lambda_1 x\}$$

and

$$(6) \quad q_1 = n_1 + n_2 + \dots + n_{m_1}.$$

Remarks. 1. The equality (4) means the fact that the number of series of eigenvectors and associated vectors, corresponding to an eigenvalue in a Jordan basis is equal to the dimension of the respective proper subspace.

2. The equality (6) means the fact that the number of vectors in a Jordan basis, corresponding to an eigenvalue is equal with the multiplicity of the eigenvalue as root of the characteristic polynomial of f .

3. From (6) it results

$$(7) \quad n_i \leq q_1, \quad i = \overline{1, m_1}$$

and

$$(7') \quad m_1 \leq q_1.$$

The inequality (7) means the fact that the number of the vectors in a series corresponding to an eigenvalue, in a Jordan basis, is less than the multiplicity of the eigenvalue as root of the characteristic polynomial.

4. From (4) and (6) it follows that a Jordan basis of the operator f contains only eigenvectors (that is every series has only one vector which is, obviously, eigenvector) if and only if, for every eigenvalue, the dimension of the corresponding proper subspace is equal with the multiplicity of the eigenvalue, as root of the characteristic polynomial.

P r o o f of Theorem 1. We prove first the assertion (A₁): Because $\{a_1^i, \dots, a_{n_i}^i\}$, ($i = \overline{1, p}$) is a series of eigenvectors and associated vectors, corresponding to λ_i , taking into account of (1'), we have

$$(8) \quad f(a_i) = \lambda_i a_i^i, \quad f(a_k^i) = \lambda_i a_k^i + a_{k-1}^i, \quad k = \overline{1, n_i}, \quad i = \overline{1, p}.$$

It follows that the matrix of the operator f , with respect to the basis (2), is

$$J = \begin{bmatrix} J_{n_1}(\lambda_1) & & 0 \\ & J_{n_2}(\lambda_2) & \\ \dots & \dots & \dots \\ 0 & & J_{n_p}(\lambda_p) \end{bmatrix}, \quad \text{where}$$

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & \lambda_i \end{bmatrix}$$

(a n_i -order Jordan block). Then, the characteristic polynomial of the operator f is

$$(9) \quad p(\lambda) = |\lambda I_n - I| = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_p)^{n_p}$$

(I_n is the n -order unit matrix). From (9) it results (A_1), the eigenvalues of f being roots of the characteristic polynomial.

We shall prove now the assertions (A_2). For this, we shall prove that the vector-system

$$(10) \quad \{a_1^1, \dots, a_1^{m_1}\}$$

is a basis in the proper subspace $S_{\lambda_1}(f)$. Because (2) is a linearly-independent vector-system, it results that (10) is linearly independent. It must still be proved that every $v \in S_{\lambda_1}(f)$ can be written as a linear combination of the vectors of (10).

Let be $v \in S_{\lambda_1}(f)$. Since (2) is a basis in X_n , v can be written in the form

$$(11) \quad v = \sum_{i=1}^{m_1} \left[\sum_{k=1}^{n_i} (\alpha_k^i a_k^i) \right] + \sum_{j=m_1+1}^p \left[\sum_{q=1}^{n_j} \alpha_q^j a_q^j \right].$$

From (11), taking into account that f is a linear operator and $\lambda_1 = \dots = \lambda_{m_1}$, it follows

$$(12) \quad f(v) = \sum_{i=1}^{m_1} \left[\alpha_1^i \lambda_1 a_1^i + \sum_{k=2}^{n_i} \alpha_k^i (\lambda_1 a_k^i + a_{k-1}^i) \right] + \sum_{j=m_1+1}^p \left[\alpha_1^j \lambda_j a_1^j + \sum_{q=2}^{n_j} \alpha_q^j (\lambda_j a_q^j + a_{q-1}^j) \right] =$$

$$= \sum_{i=1}^{m_1} \left[\lambda_1 \left[\sum_{k=1}^{n_i} \alpha_k^i a_k^i \right] + \sum_{k=2}^{n_i} \alpha_k^i a_{k-1}^i \right] + \sum_{j=m_1+1}^p \left[\lambda_j \left[\sum_{q=1}^{n_j} \alpha_q^j a_q^j \right] + \sum_{q=2}^{n_j} \alpha_q^j a_{q-1}^j \right].$$

Because $f(v) = \lambda_1(v)$, using (11) and (12), we obtain

$$\sum_{i=1}^{m_1} \left[\sum_{k=2}^{n_i} \alpha_k^i a_{k-1}^i \right] + \sum_{j=m_1+1}^p \left[\sum_{q=1}^{n_j} (\lambda_j - \lambda_1) \alpha_q^j a_q^j + \sum_{q=2}^{n_j} \alpha_q^j a_{q-1}^j \right] = 0$$

and then

$$(13) \quad \sum_{i=1}^{m_1} \left[\sum_{k=2}^{n_i} \alpha_k^i a_{k-1}^i \right] + \sum_{j=m_1+1}^p \left[\left[\sum_{q=1}^{n_j-1} (\lambda_j - \lambda_1) \alpha_q^j + \alpha_{q+1}^j \right] a_q^j + (\lambda_j - \lambda_1) \alpha_{n_j}^j a_{n_j}^j \right] = 0.$$

From (13), because the vector-sistem (2) is linearly independent, it follows

$$(14) \quad \alpha_k^i = 0, \quad k = \overline{2, n_i}, \quad i = \overline{1, m_1};$$

$$(15) \quad (\lambda_j - \lambda_1) \alpha_q^j + \alpha_{q+1}^j = 0, \quad q = \overline{1, n_j-1}, \quad j = \overline{m_1+1, p};$$

$$(16) \quad (\lambda_j - \lambda_1) \alpha_{n_j}^j = 0, \quad j = \overline{m_1+1, p}.$$

Since we have (3), from (16) it results $\alpha_{n_j}^j = 0, j = \overline{m_1+1, p}$, and then, from (15) $\alpha_q^j = 0, q = \overline{1, n_j-1}, j = \overline{m_1+1, p}$.

So, we have proved, starting of (11), that

$$(17) \quad \alpha_k^i = 0, \quad k = \overline{2, n_i}, \quad i = \overline{1, m_1}; \quad \alpha_q^j = 0, \quad q = \overline{1, n_j}, \quad j = \overline{m_1+1, p}.$$

Replacing (17) in (11) we obtain v as a linear combination of the vectors of the system (10).

The equality (6) results from (9), taking into account (3).

2. Series of Eigenvectors and Associated Vectors with the First Vectors Linearly Independent; Formation of a Jordan Basis. Theorem 2. Let $f: X_n \rightarrow X_n$ be a linear operator.

If $\{v_1^j, v_2^j, \dots, v_{n_j}^j\}, j = \overline{1, p}$, are series of eigenvectors and associated vectors corresponding to the eigenvalues $\lambda_j, j = \overline{1, p}$, of the f , and

(H) the vectors $v_1^1, v_1^2, \dots, v_1^p$ are linearly independent, then, for every $q \in \{1, 2, \dots, p\}$ the vector-system

$$(18) \quad \{v_1^1, v_2^1, \dots, v_{n_1}^1, v_1^2, v_2^2, \dots, v_{n_2}^2, \dots, v_1^q, v_2^q, \dots, v_{n_q}^q, v_1^{q+1}, \dots, v_1^p\},$$

is linearly independent.

P r o o f. The proof is similar to that of the Theorem 1 of [6] (by induction with respect to q and for $q = 1$ by induction with respect to u_1).

An immediate consequence of the Theorems 1 and 2 is the possibility of a direct determination of a Jordan basis corresponding to a linear operator $f: X_n \rightarrow X_n$. For this purpose we consider a basis $\{e_1, \dots, e_n\}$ in X_n and we find the matrix A of f with respect to this basis. Then we determine the characteristic polynomial $p(\lambda) = |\lambda I_n - A|$ and all roots, $\lambda_1, \lambda_2, \dots, \lambda_r, \lambda_i \neq \lambda_j, i \neq j, i, j = \overline{1, r}$, of $p(\lambda)$.

If $\lambda_1, \lambda_2, \dots, \lambda_r \in K$, a Jordan basis in X_n of f exists.

We determine the proper subspaces

$$S_{\lambda_i}(f) = \{x \in X_n, x = x_1 e_1 + \dots + x_n e_n, f(x) = \lambda_i x\}, \quad i = \overline{1, r}$$

and

$$m_i = \dim S_{\lambda_i}(f).$$

Then, for every λ_i ($i = \overline{1, r}$) we determine directly, by solving the suitable linear algebraic systems, m_i series of eigenvectors and associated vectors, with the first vectors of every series linearly independent and so that the number of all vectors of these m_i series is equal to the multiplicity of λ_i as root of the characteristic polynomial $p(\lambda)$.

If we consider, now, a vector-system formed with the series thus determined, as a consequence of Theorem 2, this vector system is a basis in X_n and hence a Jordan basis in X_n , of f .

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PROPRIETĂȚI PRIVIND O METODĂ DIRECTĂ DE DETERMINARE A UNEI BAZE JORDAN A UNUI OPERATOR LINIAR

(Rezumat)

Pentru un operator liniar $f: X_n \rightarrow X_n$, în care X_n este un spațiu vectorial n -dimensional peste un corp comutativ K , se demonstrează proprietăți privind seriile de vectori proprii și asociați de forma (1) (Teoremele 1 și 2). Aceste proprietăți sunt utile pentru a determina direct, rezolvând sisteme algebrice corespunzătoare, unele serii de vectori proprii și asociați, cu care se poate forma o bază Jordan în X_n , a operatorului f .

REMARKS ABOUT SOME MEAN VALUE THEOREMS

BY

GH. BANTAȘ and M. TURINICI

0. Introduction. Let I be a real interval and $(X, \|\cdot\|)$ a normed space. Take a function $f: I \rightarrow X$ and put, for each couple, $t, s \in I, t < s$

$$(Q) \quad R_f(t, s) = \frac{f(t) - f(s)}{t - s}, \quad S_f(t, s) = \|R_f(t, s)\|.$$

These will be referred to as the incrementary quotients of f with respect to (t, s) . Let also $a, b \in I, a < b$. By a mean value property of f over $[a, b]$ we usually mean an evaluation of $R_f(a, b)$ or $S_f(a, b)$ with the aid of some expressions depending on the objective to be attained. For example, a basic mean value property (or equality form) established in 1964 by McLeod [10] asserts that

$$(E) \quad R_f(a, b) \in \overline{\text{co}} \{f'(t); t \in]a, b[\setminus A\},$$

where A is a denumerable subset of $[a, b]$. As a particular case of it we have the mean value property (of inequality form)

$$(I) \quad S_f(a, b) \leq \|f'(c)\|, \quad \text{for some } c \in]a, b[\setminus A,$$

established (under the particular choice $A = \emptyset$) in the 1963 paper by Aziz and Diaz [1].

It is our main aim in the following to show that, an appropriate refinement of the authors' argument yields an extension of this last result; details will be given in Sec. 1. We also show (in Sec. 2) that a further enlargement of the obtained facts is possible by allowing exceptional (denumerable) sets. This may be viewed as a tool in proving new results of this type, comparable with the basic mean value theorem (of inequality form) due to Dieudonné [6, ch. VIII, §5],

$$(M) \quad \|f'(t)\| \leq g'(t), \quad t \in]a, b[\setminus A \Rightarrow \|f(b) - f(a)\| \leq g(b) - g(a),$$

where $g: I \rightarrow \mathbb{R}$ is increasing and A is a denumerable subset of $[a, b]$. (These aspects will be treated in a future paper.) Finally, Sec. 3 treats the case $X = \mathbb{R}$, under the lines in Pompeiu [14], [15]. This, in turn, yields a mean value property (of equality form) similar to the one in Lebourg [9].

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suggestions.

1. Main Results. Let I be a real interval and X , a normed space. We take a function $f: I \rightarrow X$ and fix two points $a, b \in I$, $a < b$. The notations (Q) are being maintained.

Theorem 1. Let $\varepsilon > 0$ be given. Then, there exists a subinterval $[a_\varepsilon, b_\varepsilon]$ of $[a, b]$ with

$$0 < b_\varepsilon - a_\varepsilon < \varepsilon, \quad S_f(a, b) \leq S_f(a_\varepsilon, b_\varepsilon).$$

Proof. Let us construct a division $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ of the interval $[a, b]$ with $t_{i+1} - t_i < \varepsilon$, $0 < i < n - 1$ (where n is a positive integer). It is clear that

$$R_f(a, b) = \lambda_0 R_f(t_0, t_1) + \dots + \lambda_{n-1} R_f(t_{n-1}, t_n),$$

where, by convection,

$$\lambda_i = \frac{t_{i+1} - t_i}{b - a}, \quad 0 \leq i \leq n - 1.$$

Therefore (by the triangle inequality)

$$(1.1) \quad S_f(a, b) \leq \lambda_0 S_f(t_0, t_1) + \dots + \lambda_{n-1} S_f(t_{n-1}, t_n).$$

Now, the second member of (1.1) is a convex combination of $S_f(t_0, t_1), \dots, S_f(t_{n-1}, t_n)$. Hence the conclusion. Q.E.D.

It is to be underlined that no property is required for the function f to get the property in this statement. Nevertheless, the obtained conclusion is not very sharp; because the possibility $a_\varepsilon = a$ or $b_\varepsilon = b$ cannot be avoided, in general. It is therefore natural to ask under which conditions is this possible. An appropriate answer is concentrated in

Theorem 2. Suppose that

(C₀) f is continuous at a and b .

Then, for each $\varepsilon > 0$, there exists a subinterval $[a_\varepsilon, b_\varepsilon]$ of $]a, b[$ with the properties

$$0 < b_\varepsilon - a_\varepsilon < \varepsilon, \quad S_f(a, b) \leq S_f(a_\varepsilon, b_\varepsilon).$$

Proof. Let the division of $[a, b]$ be constructed as in Theorem 1. Relation (1.1) gives

$$(1.2) \quad S_f(a, b) \leq \max\{S_f(t_0, t_1), \dots, S_f(t_{n-1}, t_n)\}.$$

Two cases are open before us:

Case 1. Relation (1.2) is holding strictly (with $<$ in place of $\leq$). If

$$\max\{S_f(t_0, t_1), \dots, S_f(t_{n-1}, t_n)\} = S_f(t_i, t_{i+1})$$

for some $i \in \{1, \dots, n - 2\}$ then, we are done, by putting $a_\varepsilon = t_i$, $b_\varepsilon = t_{i+1}$. If, however,

$$\max\{S_f(t_0, t_1), \dots, S_f(t_{n-1}, t_n)\} = S_f(a, t_1) \quad \text{or} \quad S_f(t_{n-1}, b),$$

then, one proceeds as follows. One has $S_f(a, b) < S_f(a, t_1)$ or $S_f(t_{n-1}, b)$. The mapping $\tau \mapsto S_f(\tau, t_1)$ (resp., $\tau \mapsto S_f(t_{n-1}, \tau)$) is, by (C₀), continuous at $\tau = a$ (resp., $\tau = b$). Hence $S_f(a, b) < S_f(\sigma, t_1)$ or $S_f(t_{n-1}, \sigma)$ for some $\sigma \in]a, t_1[$ (resp., $]t_{n-1}, b[$). And then, putting $a_\varepsilon = \sigma$, $b_\varepsilon = t_1$ (resp., $a_\varepsilon = t_{n-1}$, $b_\varepsilon = \sigma$) we obtain the desired conclusion.

Case 2. Relation (1.2) holds with equality. Then, again combining with

(1.1), $S_f(a, b) = \lambda_0 S_f(t_0, t_1) + \dots + \lambda_{n-1} S_f(t_{n-1}, t_n)$, wherefrom

$$\lambda_0 (S_f(a, b) - S_f(t_0, t_1)) + \dots + \lambda_{n-1} (S_f(a, b) - S_f(t_{n-1}, t_n)) = 0.$$

But, $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ are strictly positive. Therefore $S_f(a, b) = S_f(t_0, t_1) = \dots = S_f(t_{n-1}, t_n)$ and, from this, the conclusion is clear. Q.E.D.

As a direct consequence, we have the following fact established by A z i z and D i a z [1]:

C o r o l l a r y 1. Suppose that
(C^S) f is (strongly) continuous over $[a, b]$.

Then, for each $\varepsilon > 0$, there is a subinterval $[a_\varepsilon, b_\varepsilon]$ of $]a, b[$ for which conclusion of Theorem 2 is retainable.

Remark. Condition (C₀) includes effectively (C^S); hence this corollary is strictly included in Theorem 2.

The obtained results are of relative character; that is, the incrementary quotient over $[a, b]$ is compared with an object of the same type over a subinterval $[a_\varepsilon, b_\varepsilon]$ of it. It is natural to ask what happens when, roughly speaking, ε tends to zero in these relations. The answer is contained in

T h e o r e m 3. Suppose that f satisfies condition (C₀) in Theorem 2. Then, there exists an element c in $]a, b[$, with

$$S_f(a, b) \leq \max(\limsup_{\tau \rightarrow c^-} S_f(\tau, c), \limsup_{\tau \rightarrow c^+} S_f(c, \tau)).$$

P r o o f. By Theorem 2, we have promised a subinterval $[a_1, b_1]$ of $]a, b[$, with $0 < b_1 - a_1 \leq 2^{-1}(b - a)$, $S_f(a, b) \leq S_f(a_1, b_1)$. Further, by Theorem 1, we determine a subinterval $[a_2, b_2]$ of $[a_1, b_1]$ with $0 < b_2 - a_2 \leq 2^{-1}(b_1 - a_1)$, $S_f(a_1, b_1) \leq S_f(a_2, b_2)$. (Note that, the alternatives $a_2 = a_1$ or $b_2 = b_1$ cannot be excluded.) This process (of repeatedly applying Theorem 1) can be continued. We therefore construct a sequence of subintervals $([a_n, b_n])_{n \geq 1}$ of $]a, b[$, with

$$(1.3) \quad]a, b[\supseteq [a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots,$$

$$(1.4) \quad b_n - a_n \leq 2^{-n}(b - a), \quad n \geq 1,$$

$$(1.5) \quad S_f(a, b) \leq S_f(a_1, b_1) \leq S_f(a_2, b_2) \leq \dots$$

By (1.3) + (1.4) we have (via Cantor's intersection theorem) that there must be a (single) point $c \in]a, b[$ with

$$(1.6) \quad a_n \leq c \leq b_n, \quad n \geq 1,$$

$$(1.7) \quad a_n \rightarrow c, \quad b_n \rightarrow c \quad \text{as} \quad n \rightarrow \infty.$$

We have several situations to discuss.

C a s e 1. The sequence (a_n) is stationary. That is (combining with (1.6)) $a_n = c < b_n$, for all n . (Actually, such a relation may be valid for all $n \geq n_0$ only, where n_0 is large enough. But, this is not a restriction.) In view of (1.5),

$$(1.8) \quad S_f(a, b) \leq S_f(c, b_n), \quad n \geq 1.$$

Hence, taking (1.7) into account

$$S_f(a, b) \leq \limsup_{\tau \rightarrow c^+} S_f(c, \tau),$$

and conclusion follows.

C a s e 2. The sequence (b_n) is stationary. That is (combining with (1.6)) $a_n < c = b_n$, for all n . So, by (1.5) again

$$(1.9) \quad S_f(a, b) \leq S_f(a_n, c), \quad n \geq 1.$$

And then, from (1.7),

$$S_f(a, b) \leq \limsup_{\tau \rightarrow c^-} S_f(\tau, c),$$

again proving the desired conclusion.

C a s e 3. Both sequences (a_n) and (b_n) are non-stationary. Without loss, one may assume (a_n) and (b_n) are strictly monotone; or, in other words, inequalities in (1.6) are strict. From $R_f(a_n, b_n) = \alpha R_f(a_n, c) + \beta R_f(c, b_n)$, where, by convection,

$$\alpha = \frac{c - a_n}{b_n - a_n}, \quad \beta = \frac{b_n - c}{b_n - a_n},$$

one gets, by the triangle inequality,

$$S_f(a_n, b_n) \leq \alpha S_f(a_n, c) + \beta S_f(c, b_n) \leq \max\{S_f(a_n, c), S_f(c, b_n)\}.$$

Hence, passing eventually to a subsequence, one of the relations (1.8) or (1.9) holds true. And from this, conclusion follows in the way we already presented. Q.E.D.

Remark. In view of (C_0) conclusion of Theorem 3 is retainable even if the function f is discontinuous at some points of $]a, b[$. However, in such a case it becomes trivial. Because, if the underlying function is discontinuous at some $c \in]a, b[$, then

$$\max(\limsup_{\tau \rightarrow c^-} \|f(\tau) - f(c)\|, \limsup_{\tau \rightarrow c^+} \|f(\tau) - f(c)\|) > 0,$$

wherefrom

$$\max(\limsup_{\tau \rightarrow c^-} S_f(\tau, c), \limsup_{\tau \rightarrow c^+} S_f(c, \tau)) = \infty,$$

if we take into account some well known properties of the "lim sup" operator (see, e.g., M e g h e a [11, ch. IX, § 2]).

It follows that the natural framework of Theorem 3 is that of (C^S) being accepted in place of (C_0) . But then, conclusion above may be improved. Specifically, one has

T h e o r e m 4. Suppose that f satisfies condition (C^S) in Corollary 1. Then, there exists $c \in]a, b[$ such that

$$S_f(a, b) \leq \limsup_{\substack{\alpha, \beta \rightarrow c \\ \alpha < c < \beta}} S_f(\alpha, \beta).$$

P r o o f. By repeatedly applying Theorem 1 we may construct a sequence of subintervals $([a_n, b_n])_{n \geq 1}$ of $]a, b[$, with the properties (1.4), (1.5) plus

$$(1.3)' \quad]a_n, b_n[\supseteq]a_{n+1}, b_{n+1}[, \quad \text{for all } n.$$

Now, by the Cantor intersection theorem, there must be a (single) point $c \in]a, b[$, with the property (1.7) plus

$$(1.6)' \quad a_n < c < b_n, \quad n \geq 1.$$

Hence, Case 3 of the above discussion is attained. And, from this, we end the argument. Q.E.D.

Remark. In view of

$$S_f(\alpha, \beta) \leq \max(S_f(\alpha, c), S_f(c, \beta)), \quad \alpha < c < \beta$$

it follows that the evaluation of the conclusion above is sharper than the one appearing in Theorem 3. This may become important for those situations discussed in the Remark above. That is, it might very well happen that the right member of the above evaluation be finite in that point c of $]a, b[$ at which f is discontinuous.

Another improvement of Theorem 3 is to be obtained under the line in Diaz and Vybony [5]. More exactly

Theorem 5. Suppose that f satisfies condition (C^S) in Corollary 1. Then, for at least one c in $]a, b[$, it is the case that

$$S_f(a, b) \leq \max(\liminf_{\tau \rightarrow c^-} S_f(\tau, c), \liminf_{\tau \rightarrow c^+} S_f(c, \tau)).$$

Proof (Sketch). Denote

$$\varphi(t) = \|f(t) - f(a)\| - (t-a)S_f(a, b), \quad t \in I.$$

By (C^S) , φ is continuous over $[a, b]$. In addition, $\varphi(a) = \varphi(b) = 0$. By the Weierstrass theorem, there must be at least one c in $]a, b[$ with either $\varphi(c) \geq \varphi(t)$, $a \leq t \leq b$ or $\varphi(c) \leq \varphi(t)$, $a < t < b$. To make a choice, assume that the former of these holds. We therefore have

$$\|f(c) - f(a)\| - \|f(t) - f(a)\| \geq (c-t)S_f(a, b), \quad a \leq t < c.$$

As a consequence, $S_f(a, b) \leq S_f(\tau, c)$, $a \leq \tau < c$, wherefrom conclusion is clear. The remaining situation is treatable in a similar way. Hence the result. Q.E.D.

Remark. Call the function f , quasicontinuous at the point $c \in]a, b[$, in case

$$\max(\liminf_{\tau \rightarrow c^-} \|f(\tau) - f(c)\|, \liminf_{\tau \rightarrow c^+} \|f(\tau) - f(c)\|) = 0.$$

Now, conclusion of Theorem 5 is trivial whenever f is quasi-discontinuous at some $c \in]a, b[$. For, in such a case (cf. the reasoning we already presented)

$$\max(\liminf_{\tau \rightarrow c^-} S_f(\tau, c), \liminf_{\tau \rightarrow c^+} S_f(c, \tau)) = \infty.$$

In other words, a natural framework of Theorem 5 is that of (H_1^S) being replaced by (C_0) plus

(C^*) f is quasi-continuous on $]a, b[$.

Some further aspects will be discussed elsewhere.

As an application of this result, one has (cf. Aziz and Diaz [1])

Corollary 2. Let the function f be such that, condition (H_1^S) of Corollary 1, as well as

(D_0^S) f is (strongly) derivable over $]a, b[$
are holding. Then,

$$S_f(a, b) \leq \|f'(c)\|, \quad \text{for some } c \in]a, b[.$$

2. Some Extensions. A close analysis of the arguments above shows that only a part of the working framework is effectively needed in deducing these conclusions. In other words, structural extensions of the preceding statements are possible. These may be obtained according to the following

Remarks. 1. In the reasonings above, the symmetry of the norm were not effectively used. So, these statements remain valid if the norm $x \mapsto \|x\|$ of the ambient space is to be substituted by a sublinear continuous function $x \mapsto p(x)$ from X to $[0, \infty)$. Some interesting aspects may be found in Flett [7, ch. I, § 6].

2. Throughout those developments, the algebraic properties of the real numbers in $[a, b]$ were ignored. Hence, if the interval in question is replaced by a metrical segment (in the sense of Blumenthal and Menger [4, ch. IV, § 3]), the argument above will be not affected. We refer to the paper by Turinici [16] for some technical aspects.

3. The total ordering in $[a, b]$ may be removed, if we interpret f as a set function, i.e., $f(t) = f([a, t])$, $a \leq t \leq b$. In this case, the interval $[a, b]$ may be replaced, e.g., by a rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$. See Niculescu [13] for a result of this kind.

Now, an interesting way of extension is by allowing exceptional sets. This, evidently, refers to the statements involving the intermediary point c . Specifically, a useful extension of Theorem 3 is

Theorem 3'. Suppose that f satisfies

(C_1) f is continuous over $A \cup \{a, b\}$,

where A is a denumerable part of $[a, b]$ with a finite number of accumulation points. Then, there must be an element c in $]a, b[\setminus A$, with the property stated in Theorem 3.

Proof. Let $d_0, d_1, \dots, d_n$ be the set of all accumulation points of A . Without loss one may assume $a = d_0 < d_1 < \dots < d_{n-1} < d_n = b$. By Theorem 1, $S_f(a, b) \leq S_f(d_k, d_{k+1})$, for some $k \in \{0, 1, \dots, n-1\}$. Further, by (C_1) plus Theorem 2, there exists a closed subinterval $[d'_k, d'_{k+1}]$ of $]d_k, d_{k+1}[$ with $S_f(d'_k, d'_{k+1}) \leq S_f(d_k, d_{k+1})$.

Now, by the choice of A , the intersection $[d'_k, d'_{k+1}] \cap A$ is necessarily finite; denote it as $\{e_k^0, \dots, e_k^m\}$. Of course, again without loss, one may assume

$$d'_k = e_k^0 < e_k^1 < \dots < e_k^m = d'_{k+1}.$$

Again by Theorem 1, $S_f(d'_k, d'_{k+1}) \leq S_f(e_k^j, e_k^{j+1})$, for some $j \in \{0, 1, \dots, m-1\}$. We now apply Theorem 3 to the interval $[e_k^j, e_k^{j+1}]$; one obtains a point c in $]e_k^j, e_k^{j+1}[$ (hence not belonging to A) with

$$S_f(e_k^j, e_k^{j+1}) \leq \max(\limsup_{\tau \rightarrow c^-} S_f(\tau, c), \limsup_{\tau \rightarrow c^+} S_f(c, \tau)).$$

This, plus the inequalities above, ends the argument. Q.E.D.

Similarly, an appropriate refinement of Theorem 4 is

Theorem 4'. Suppose that f satisfies condition (C^S) in Corollary 1. Let also A be a denumerable part of $[a, b]$ with a finite number of accumulation points. Then, there exists a $c \in]a, b[\setminus A$ with the property stated in Theorem 4.

Now, an extension of Theorem 5 under the lines above is immediate; so, we do not give details. Finally, Corollary 2 has the following enlargement:

Corollary 2'. Let the function f be such that condition (C^S) of Corollary 1, as well as

(D_1^S)

f is (strongly) derivable over $]a, b[\setminus A$

are holding, where A is a denumerable part of $[a, b]$ with a finite number of accumulation points. Then

$$S_f(a, b) \leq \|f'(c)\|, \quad \text{for some } c \in]a, b[\setminus A.$$

Of course, all these statements have structural extensions constructed under the lines precised in Remarks 1, 2, 3 at the beginning of this section. For example, an extension of Corollary 2' under Remark 1 yields a result comparable with the one in McLeod [10].

Finally, a common way of extension is the following. Let $g: I \rightarrow \mathbb{R}$ be a strictly increasing function. Denote, for each couple $t, s \in I, t < s$

$$(Q') \quad R_{f,g}(t, s) = \frac{f(t) - f(s)}{g(t) - g(s)}, \quad S_{f,g}(t, s) = \|R_{f,g}(t, s)\|.$$

A remarkable fact we have to underline here is that, under the same hypotheses about f as before, the stated results will remain valid with $S_{f,g}$ in place of S_f . This seems to be without importance from a technical perspective. We shall treat these aspects in a forthcoming paper.

3. The Real Case. In the following, the case $X = \mathbb{R}$ will be analyzed, from an equality perspective. So, let I be a real interval and $f: I \rightarrow \mathbb{R}$, a function. Fix also $a, b \in I, a < b$. The notations (Q) (in the introductory part) are in use. As a counterpart of Theorem 2, we have

Theorem 6. Suppose that f satisfies condition (C^S) of Corollary 1. Then, for each $\varepsilon > 0$, there is a subinterval $[a_\varepsilon, b_\varepsilon]$ of $]a, b[$ with

$$0 < b_\varepsilon - a_\varepsilon < \varepsilon, \quad R_f(a, b) = R_f(a_\varepsilon, b_\varepsilon).$$

Proof. Let $n \geq 3$ be a positive integer, and $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ be an equi-distant division of $[a, b]$ with norm inferior to ε , that is, $t_{i+1} - t_i = \omega < \varepsilon, 0 \leq i \leq n-1$. We introduce the function $\varphi(t) = R_f(t, t + \omega), a \leq t \leq t_{n-1}$.

It is clear that

$$(3.1) \quad R_f(a, b) = \lambda_0 \varphi(t_0) + \dots + \lambda_{n-1} \varphi(t_{n-1}),$$

where $(\lambda_i; 0 \leq i \leq n-1)$ are introduced as in Theorem 1. Two cases are open before us:

Case 1. The set $\{\varphi(t_0), \dots, \varphi(t_{n-1})\}$ consists of exactly one element. As a

consequence $R_f(a, b) = R_f(t_1, t_2)$ and conclusion is clear (with $a_\varepsilon = t_1$, $b_\varepsilon = t_2$).

C a s e 2. The set $\{\varphi(t_0), \dots, \varphi(t_{n-1})\}$ has at least two distinct elements. Hence

$$(3.2) \quad \min\{\varphi(t_0), \dots, \varphi(t_{n-1})\} < \max\{\varphi(t_0), \dots, \varphi(t_{n-1})\}.$$

On the other hand, by convexity arguments,

$$(3.3) \quad \min\{\varphi(t_0), \dots, \varphi(t_{n-1})\} \leq R_f(a, b) \leq \max\{\varphi(t_0), \dots, \varphi(t_{n-1})\}.$$

Suppose one of those relations holds with equality: e.g., the second. We have, by (3.1),

$$\lambda_0(R_f(a, b) - \varphi(t_0)) + \dots + \lambda_{n-1}(R_f(a, b) - \varphi(t_{n-1})) = 0.$$

As $\lambda_0, \dots, \lambda_{n-1}$ are strictly positive, $R_f(a, b) = \varphi(t_0) = \dots = \varphi(t_{n-1})$, absurd by (3.2). So both inequalities in (3.3) are strict. Suppose

$$\min\{\varphi(t_0), \dots, \varphi(t_{n-1})\} = \varphi(t_i), \quad \max\{\varphi(t_0), \dots, \varphi(t_{n-1})\} = \varphi(t_j)$$

for some $i, j \in \{0, \dots, n-1\}$, $i \neq j$; of course, without loss one may assume $i < j$ (hence $t_i < t_j$). So

$$(3.4) \quad \varphi(t_i) < R_f(a, b) < \varphi(t_j).$$

By (C^S) , φ is continuous over its existence domain; and therefore it has the Darboux property over $[t_i, t_j]$. This, in view of (3.4) shows that, for some $r \in]t_i, t_j[\subseteq]a, b[$, it is the case that $R_f(a, b) = \varphi(r) = R_f(r, r+\omega)$. And then, putting $a_\varepsilon = r$, $b_\varepsilon = r+\omega$, one gets the desired conclusion. Q.E.D.

Remark. The idea of the argument is due to Pompeiu [14], [15]. A completion of Pompeiu's reasoning was performed in Aziz and Diaz [3]. Hence, this result is but a refinement of these techniques.

As an application of Theorem 6, one gets the following "weak" counterpart of Corollary 1. Let I be a real interval, X , a normed space and $f: I \rightarrow X$, a function. Let us also fix $a, b \in I$ with $a < b$. We have (cf. also Aziz and Diaz [2])

C o r o l l a r y 3. Suppose that

$$(C^W) \quad f \text{ is weakly continuous over } [a, b].$$

Then, for each $\varepsilon > 0$, there is a subinterval $[a_\varepsilon, b_\varepsilon]$ of $]a, b[$ for which conclusion of Theorem 2 is holding.

P r o o f. By the Hahn-Banach theorem, one may find a linear continuous functional x^* in X^* with $\|x^*\| = 1$, $x^*(R_f(a, b)) = S_f(a, b)$. The function $\bar{g}: I \rightarrow \mathbb{R}$, given by $\bar{g}(t) = x^*(f(t))$, $t \in I$, fulfils, by (C^W) , conditions of Theorem 6. So, for each $\varepsilon > 0$, there exists a subinterval $[a_\varepsilon, b_\varepsilon]$ of $]a, b[$, with $0 < b_\varepsilon - a_\varepsilon < \varepsilon$, $R_g(a, b) = R_g(a_\varepsilon, b_\varepsilon)$. But $R_g(a, b) = |x^*(R_f(a, b))| = S_f(a, b)$; and, moreover,

$$R_g(a_\varepsilon, b_\varepsilon) = |x^*(R_f(a_\varepsilon, b_\varepsilon))| \leq S_f(a_\varepsilon, b_\varepsilon).$$

Combining these facts yields the desired conclusion. Q.E.D.

Now, Theorem 6 is also useful in deducing an interesting counterpart of Theorem 3. This may be done as follows. By an extended gradient of the function $f: I \rightarrow \mathbb{R}$ at the point $t \in]a, b[$, we mean the subset $\partial^e f(t) \subseteq [-\infty, \infty]$ for all extended real numbers x with

$$\liminf_{\substack{r, s \rightarrow t \\ r < t < s}} R_f(r, s) \leq x \leq \limsup_{\substack{r, s \rightarrow t \\ r < t < s}} R_f(r, s).$$

Note that, under the acceptance of (H_2^S) , $\partial^e f(t) = \{f'(t)\}$, $a < t < b$; hence, the term "generalized" is motivated.

Theorem 7. Suppose that f satisfies condition (C^S) of Corollary 1.

Then

$$R_f(a, b) \in \partial^e f(c), \quad \text{for some } c \in]a, b[.$$

Proof. By Theorem 6, we have promised a sequence of subintervals $([a_n, b_n])_{n \geq 1}$ of $]a, b[$ with

$$(3.4) \quad]a_n, b_n[\supseteq]a_{n+1}, b_{n+1}[, \quad \text{for all } n,$$

$$(3.5) \quad b_n - a_n = 0, \quad \text{as } n \rightarrow \infty,$$

$$(3.6) \quad R_f(a, b) = R_f(a_n, b_n), \quad \text{for all } n.$$

Now, (3.4)+(3.5) yields (via Cantor's intersection theorem) a point $c \in]a, b[$ with the property (1.7) plus (1.6). Combining these informations give

$$\liminf_{\substack{r, s \rightarrow c \\ r < c < s}} R_f(r, s) \leq R_f(a, b) \leq \limsup_{\substack{r, s \rightarrow c \\ r < c < s}} R_f(r, s).$$

This completes the argument. Q.E.D.

The obtained result is comparable to the one given in Lebourg [9]. Further aspects of the problem may be found in the survey papers by Nashed [12] and Hiriart-Urruty [8].

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OBSERVAȚII ASUPRA UNOR TEOREME DE VALOARE MEDIE

(Rezumat)

Se deduc rezultatele privind valoarea medie ale lui Aziz și Diaz, Nashed și McLeod, utilizând ipoteze minime pentru funcția vectorială. De asemenea, pe o direcție dezvoltată de Pompeiu, se rafinează o teoremă în același sens, datorată lui Aziz și Diaz.

POINTS FIXES COMMUNS POUR DES APPLICATIONS EXPANSIVES

PAR

NICOLETA NEGOESCU

1. Dans quelques Notes récentes [1], [3], [7] plusieurs auteurs ont établi d'intéressants théorèmes de point fixe pour une ou deux autoapplications expansives définies sur des espace métriques complets ou sur des espace métriques compacts qui sont des analogues pour certains théorèmes de point fixe pour une ou deux autoapplications contractives [6].

Le but de cette Note est de démontrer quelques théorèmes de point fixe pour deux autoapplications expansives qui satisfont aux conditions d'un autre type que celles des notes citées, conditions que nous ont été suggérées par les théorèmes de point fixe pour des applications contractives obtenus par Jain et Dixit [4] et Nicoleta Negoescu [5].

2. *Théorème 1. Soient (X, d) un espace métrique complet et $S, T: X \rightarrow X$ deux opérateurs surjectifs ($S(X)=X$ et $T(X)=X$) qui satisfont à l'inégalité*

$$(1) \quad d^2(Sx, Ty) \geq ad^2(x, Sx) + bd^2(y, Ty) + cd(x, Ty)d(y, Sx)$$

pour tous $x, y \in X$, $x \neq y$, où $a+b > 1$, $0 < a < 1$, $b > 0$, $c \geq 0$.

Alors S et T ont un point fixe commun $u \in X$, qui pour $c > 1$ est un point fixe unique.

Démonstration. Soit x_0 un point arbitraire de X . Nous définissons la suite (x_n) de la manière suivante:

$$x_0 = Sx_1, \quad x_1 = Tx_2, \quad \dots, \quad x_{2n} = Sx_{2n+1}, \quad x_{2n+1} = Tx_{2n+2}, \dots$$

Nous supposons que $x_{2n} = x_{2n+1}$ pour un $n \in \mathbb{N}$. Si $x_{2n+1} = x_{2n+2}$ alors en employant la condition (1) nous obtenons

$$\begin{aligned} d^2(x_{2n}, x_{2n+1}) &= d^2(Sx_{2n+1}, Tx_{2n+2}) \geq ad^2(x_{2n+1}, Sx_{2n+1}) + \\ &+ bd^2(x_{2n+2}, Tx_{2n+2}) + cd(x_{2n+1}, Tx_{2n+2})d(x_{2n+2}, Sx_{2n+1}) = \\ &= ad^2(x_{2n+1}, x_{2n}) + bd^2(x_{2n+2}, x_{2n+1}) + cd(x_{2n+1}, x_{2n+2})d(x_{2n+2}, x_{2n}), \end{aligned}$$

donc $d^2(x_{2n}, x_{2n+1}) \geq bd^2(x_{2n+1}, x_{2n+2})$ ce qui est une contradiction ($b > 0$). Alors nous avons que $x_{2n} = x_{2n+1} = x_{2n+2}$.

On peut montrer dans la même manière que si $x_{2n+1} = x_{2n+2}$ pour un $n \in \mathbb{N}$ alors $x_{2n+1} = x_{2n+2} = x_{2n+3}$.

Les deux affirmations antérieures montrent que les opérateurs S et T ont un point fixe commun (dans les cas considérés).

Nous supposons maintenant que $x_{2n+1} \neq x_{2n+2} \neq x_{2n+3}$ pour tous $n \geq 0$. Alors en employant la condition (1) il suit

$$d(x_{2n}, x_{2n+1}) > 0,$$

$$\begin{aligned} d^2(x_{2n}, x_{2n+1}) &= d^2(Sx_{2n+1}, Tx_{2n+2}) \geq ad^2(x_{2n+1}, Sx_{2n+1}) + \\ &+ bd^2(x_{2n+2}, Tx_{2n+2}) + cd(x_{2n+1}, Tx_{2n+2})d(x_{2n+2}, Sx_{2n+1}) = \\ &= ad^2(x_{2n}, x_{2n+1}) + bd^2(x_{2n+1}, x_{2n+2}) + cd(x_{2n+1}, x_{2n+1})d(x_{2n}, x_{2n+1}), \end{aligned}$$

donc $(1 - a)d^2(x_{2n}, x_{2n+1}) \geq bd^2(x_{2n+1}, x_{2n+2})$, ou

$$d^2(x_{2n}, x_{2n+1}) \geq \frac{b}{1-a} d^2(x_{2n+1}, x_{2n+2}).$$

Aussi on obtient

$$\begin{aligned} d^2(x_{2n+1}, x_{2n+2}) &= d^2(Tx_{2n+2}, Sx_{2n+3}) \geq ad^2(x_{2n+2}, Tx_{2n+2}) + \\ &+ bd^2(x_{2n+3}, Sx_{2n+3}) + cd(x_{2n+2}, Sx_{2n+3})d(x_{2n+3}, Tx_{2n+2}) = \\ &= ad^2(x_{2n+1}, x_{2n+2}) + bd^2(x_{2n+2}, x_{2n+3}) + cd(x_{2n+2}, x_{2n+2})d(x_{2n+1}, x_{2n+3}). \end{aligned}$$

Donc $(1 - a)d^2(x_{2n+1}, x_{2n+2}) \geq bd^2(x_{2n+2}, x_{2n+3})$, ou

$$d^2(x_{2n+1}, x_{2n+2}) \geq \frac{b}{1-a} d^2(x_{2n+2}, x_{2n+3}).$$

Nous avons montré que la suite (x_n) est une suite de Cauchy ($b/(1 - a) > 1$). Mais l'espace (X, d) est complet et il suit qu'il existe un point $u \in X$ tel que $x_n \rightarrow u$. L'opérateur S est surjectif, donc il existe un point $v \in X$ tel que $Sv = u$.

Nous pouvons supposer que $v = x_{2n+2}$ pour une infinité de n (sans affecter la généralité) et alors, en utilisant (1), nous obtenons

$$\begin{aligned} d^2(u, x_{2n+1}) &= d^2(Sv, Tx_{2n+2}) \geq ad^2(v, Sv) + \\ &+ bd^2(x_{2n+2}, Tx_{2n+2}) + cd(v, Tx_{2n+2})d(x_{2n+2}, Sv). \end{aligned}$$

En faisant $n \rightarrow \infty$ dans l'inégalité précédente il suit

$$0 \geq ad^2(u, v) + bd^2(u, u) + cd(v, u)d(u, u),$$

ce qui signifie $0 \geq ad^2(u, v)$, donc $d^2(u, v) = 0$ et $u = v$ (alors $Sv = u = v$) et $Su = u$.

De la même manière il suit $Tu = u$.

Nous supposons qu'il existent deux points fixes communs u et v pour S et T , donc $Su = u$, $Sv = v$ et $Tu = u$, $Tv = v$. Alors $d^2(u, v) = d^2(Su, Tv) \geq ad^2(u, Su) + bd^2(v, Tv)$.

$Tv) + cd(u, Tv)d(v, Su)$, donc $d^2(u, v) \geq cd^2(u, v)$ ou $(1 - c)d^2(u, v) \geq 0$, et, si $c > 1$, on a obtenu une contradiction.

Donc $d^2(u, v) = 0$ pour $c > 1$. Alors les opérateurs S et T ont un point fixe commun unique.

Observation 1. On peut obtenir aussi un théorème de point fixe pour un seul opérateur si dans le Théorème 1 on pose $S = T$.

Théorème 2. Soient (X, d) un espace métrique complet, $S, T: X \rightarrow X$ deux opérateurs continus surjectifs ($S(X) = X, T(X) = X$) qui satisfont à la condition

$$(2) \quad d^2(Sx, Ty) \geq a \min \{d(x, Sx)d(x, y), d(x, y)d(y, Ty), d(x, Sx)d(y, Ty)\},$$

$$\forall x, y \in X, \text{ où } a > 1.$$

Alors les opérateurs S et T ont un point fixe commun unique.

Démonstration. Soit x_0 un point arbitraire de X . On construit la suite (x_n) de la même manière que dans le Théorème 1. En employant la condition (2) nous obtenons

$$d^2(x_{2n}, x_{2n+1}) = d^2(Sx_{2n+1}, Tx_{2n+2}) \geq a \min \{d(x_{2n+1}, Sx_{2n+1})d(x_{2n+1}, x_{2n+2}),$$

$$d(x_{2n+1}, x_{2n+2})d(x_{2n+2}, Tx_{2n+2}), d(x_{2n+1}, Sx_{2n+1})d(x_{2n+2}, Tx_{2n+2})\},$$

ou

$$d^2(x_{2n}, x_{2n+1}) \geq a \min \{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), d^2(x_{2n+1}, x_{2n+2})\}.$$

Il y a deux cas:

1) Si $\min \{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), d^2(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})$ on a $d^2(x_{2n}, x_{2n+1}) \geq ad(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})$. Donc $d(x_{2n}, x_{2n+1}) \geq ad(x_{2n+1}, x_{2n+2})$ (car $d(x_{2n+1}, x_{2n+2}) > 0$) et il suit

$$(3) \quad d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{a} d(x_{2n}, x_{2n+1}).$$

2) Si $\min \{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2}), d^2(x_{2n+1}, x_{2n+2})\} = d^2(x_{2n+1}, x_{2n+2})$ on a $d^2(x_{2n}, x_{2n+1}) \geq ad^2(x_{2n+1}, x_{2n+2})$ ou

$$(3') \quad d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{\sqrt{a}} d(x_{2n}, x_{2n+1}).$$

De la même manière on obtient les inégalités

$$(3'') \quad d(x_{2n}, x_{2n+1}) \leq \frac{1}{\sqrt{a}} d(x_{2n-1}, x_{2n}),$$

$$(3''') \quad d(x_{2n}, x_{2n+1}) \leq \frac{1}{\sqrt{a}} d(x_{2n-1}, x_{2n}).$$

Donc la suite (x_n) est une suite de Cauchy (voir les inégalités (3) – (3''')). (X, d) étant un espace métrique complet, il suit qu'il existe un point $u \in X$ tel que $\lim_{n \rightarrow \infty} x_n = u$. Mais $x_{2n} = Sx_{2n+1}$ et $x_{2n+1} = Tx_{2n+2}$ et S, T sont des opérateurs continus, donc il suit que $u = Su = Tu$.

Observation 2. Si dans le Théorème 2 on pose $S=T$ alors on obtient un théorème de point fixe pour un seul opérateur T .

Observation 3. De la même manière que celle du Théorème 2 on peut démontrer le

Théorème 3. Soient (X, d) un espace métrique complet, $S, T: X \rightarrow X$ deux opérateurs continus surjectifs ($S(X)=X, T(X)=X$) qui satisfont à la condition

$$(4) \quad \begin{aligned} & \max \{d^2(Sx, Ty), d^2(x, Sx), d^2(y, Ty)\} + \\ & + b \max \{d(x, Ty)d(x, Sx), d(x, Ty)d(y, Ty)\} > ad^2(x, y) \end{aligned}$$

pour $\forall x, y \in X, x \neq y$, où $a \geq 1, b > 0$.

Alors les opérateurs S et T ont un point fixe commun unique $u \in X$.

Si l'espace métrique (X, d) est compact nous avons démontré le résultat suivant:

Théorème 4. Soient X, T deux autoapplications, S ou T continues sur un espace métrique compact (x, d) qui satisfont aux conditions suivantes:

$$(5) \quad d^2(Sx, Ty) > \min \{d(x, y)d(x, Sx), d(x, y)d(y, Ty), d(x, Sx)d(y, Ty)\}$$

pour tous $x, y \in X, x \neq y$.

Alors S ou T a un point fixe dans l'espace X .

Démonstration. On suppose que S est une autoapplication continue. (X, d) étant un espace métrique compact, il existe un point $x_0 \in X$ tel que $d(x_0, Sx_0) = \min \{d(x, Sx): x \in X\}$. Si $Sx_0 = x_0$ la démonstration est complète. Donc on suppose que $Sx_0 \neq x_0$. Alors il existe une suite (y_n) dans X telle que pour $n \rightarrow \infty, d(y_n, Ty_n) \rightarrow \min \{d(x, Tx): x \in X\}$.

On pose $x_n = Ty_n$ pour chaque $n \in \mathbb{N}$. Si T n'a pas de points fixes, $x_n \neq y_n$. En employant la condition (5) il suit

$$d^2(Sx_n, Ty_n) > \min \{d((x_n, y_n)d(x_n, Sx_n), d(x, y)d(y_n, Ty_n), d(x_n, Sx_n)d(y_n, Ty_n)\}.$$

Évidemment $d(x_0, Sx_0) > d(y, Ty_n)$. Pour $n \rightarrow \infty$ il suit que $d(x_0, Sx_0) \geq \sup \{d(x, Tx): x \in X\}$. Mais on a aussi

$$\begin{aligned} & \sup \{d^2(x, Tx): x \in X\} \geq d^2(Sx_0, TSx_0) > \\ & > \min \{d^2(x_0, Sx_0), d(x_0, TSx_0)d(x_0, Sx_0), d(x_0, Sx_0)d(Sx_0, TSx_0)\} \end{aligned}$$

et on obtient que $\sup \{d(x, Tx): x \in X\} > d(x_0, Sx_0)$, une contradiction.

Donc T a un point fixe dans l'espace X et la démonstration du Théorème 4 est complète.

Observation 4. Si dans le Théorème 4 on pose $S=T$ alors on obtient un théorème de point fixe analogue de ce théorème pour une seule autoapplication continue.

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PUNCTE FIXE COMUNE PENTRU APLICAȚII EXPANSIVE

(Rezumat)

Se demonstrează câteva teoreme de punct fix pentru una sau două autoaplicații expansive definite pe spații metrice complete și pe spații metrice compacte care satisfac (1), (2), (4) sau (5) și care sunt analogele unor teoreme de punct fix pentru una sau două autoaplicații contractive (v. [6]).

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PUNCTE FINE COMUNE PENTRU APLICATII EXPANSIVE

(Rezumat)

Se demonstrează că punctele fine comune de puncte în puncte sunt puncte fine comune de puncte în puncte. Se demonstrează că punctele fine comune de puncte în puncte sunt puncte fine comune de puncte în puncte. Se demonstrează că punctele fine comune de puncte în puncte sunt puncte fine comune de puncte în puncte.

ON SOME NEW INEQUALITIES RELATED TO THE ZEROS OF SOLUTIONS OF CERTAIN SECOND ORDER DIFFERENTIAL EQUATIONS (II)

BY

B. G. PACHPATTE

1. Introduction. In the first part of this paper [1], we obtain results on the zeros of solutions of the second order differential equations

$$(A) \quad (r(t) |y(t)|^p |y'(t)|^{p-2} y'(t))' + q(t) |y(t)|^{2p-2} y(t) = 0,$$

$$(B) \quad (r(t) |y(t)|^{p-2} |y'(t)|^{p-2} y'(t))' + q(t) |y(t)|^{2p-4} y(t) = 0,$$

where $t \in I = [0, \infty)$ and I contains the points a and b ($a < b$); $p \geq 2$ is a real constant; the function $r: I \rightarrow \mathbb{R} = (-\infty, \infty)$ is C^1 -smooth and $r > 0$; the function $q: I \rightarrow \mathbb{R}$ is continuous.

We give now further extensions of the results established in the four theorems from [1], to other equations recently studied in [2], [3]...[5], [8] with different view points.

2. Further Extensions. Recently in [2], [3] and [8] the authors have studied the second order differential equations of the forms

$$(C) \quad (f(t, y) y')' + g(t, y) = 0,$$

$$(D) \quad (f(t, y'))' + g(t, y) = 0,$$

which occur in a number of applications. For example, a steady state temperature distribution y in a rod is governed by the differential equation (C), where $f(t, y) > 0$ is the thermal conductivity at position t and temperature y and $g(t, y)$ describes internal heat sources. The problems of existence of solutions of equations (C) and (D) are studied by the authors in [3] and [8].

Regarding the functions involved in (C) and (D), we make the following

assumptions:

- (H₁) $f(t, z)$ is continuously differentiable on $I \times \mathbb{R} \rightarrow \mathbb{R}$ and is an odd function with respect to z such that $\operatorname{sgn} f(t, z) = \operatorname{sgn} z$ and $\lim_{z \rightarrow \infty} \frac{z}{f(t, z)}$ exists finitely for all $t \in I$; so that $z/f(t, z) \leq r(t)$, where $r(t)$ is a real-valued positive continuous function defined for $t \in I$;
- (H₂) $g: I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $|g(t, z)| \leq w(t, |z|)$, where $w: I \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ = [0, \infty)$ is a continuous function such that $w(t, u) \leq w(t, v)$ for $0 \leq u \leq v$.

Results on the asymptotic behavior of solutions of variants of equations (C) and (D) have been given recently by various investigators, see [2], [4], [5] and the references given therein. Here, we are interested in obtaining results on the oscillatory behavior of solution of Eqs. (C) and (D) employing inequalities of the type (1) and (4) given in Theorems 1 and 2 from [1]. We shall restrict our consideration to those solutions of (C) and (D) which exist on I and are nontrivial.

We first establish the following result, which deals with the inequalities of the type (1) and (4) from [1] corresponding to the Eqs. (C) and (D).

Theorem 1. Assume that the hypotheses (H₁) – (H₂) hold. Let $y(t)$ be a solution of (C) (and respectively of (D)), with $y(a) = y(b) = 0$, and $y(t) \neq 0$, $t \in (a, b)$. Let c be a point in (a, b) where $|y(t)|$ is maximized. Then, corresponding to the solution of (C) (and respectively of (D)), we have

$$(1) \quad 1 \leq \frac{1}{2} \left[\int_a^b r(s) ds \right] \left[\frac{1}{M^2} \int_a^b w(s, M) ds \right],$$

and

$$(2) \quad 1 \leq \frac{1}{4} \left[\int_a^b r(s) ds \right] \left[\frac{1}{M} \int_a^b w(s, M) ds \right],$$

where $M = \max |y(t)| = |y(c)|$, $c \in (a, b)$.

We next establish the following result which gives sufficient conditions for the oscillatory solution of Eqs. (C) and (D) converging to zero as $t \rightarrow \infty$.

Theorem 2. Assume that the hypotheses (H₁) – (H₂) hold and

$$(3) \quad \int_a^\infty r(s) ds < \infty.$$

i) Suppose that there exists a continuous function $h_1: I \rightarrow \mathbb{R}_+$ such that

$$(4) \quad \frac{1}{z^2} w(t, z) \leq h_1(t),$$

for all large $z > 0$ and

$$(5) \quad \int_0^{\infty} h_1(s) ds < \infty.$$

Then every oscillatory solution of (C) converges to zero as $t \rightarrow \infty$.

ii) Suppose that there exists a continuous function $h_2: I \rightarrow \mathbb{R}_+$ such that

$$(6) \quad \frac{1}{z} w(t, z) \leq h_2(t),$$

for all large $z > 0$ and

$$(7) \quad \int_0^{\infty} h_2(s) ds < \infty.$$

Then every oscillatory solution of (D) converges to zero as $t \rightarrow \infty$.

In order to prove the inequality (1) in Theorem 1, let $y(t)$ be a solution of (C). As in the proof of Theorem 1 from [1], from the hypotheses we have the relations (7) and (8) from [1], implying

$$(8) \quad M^2 \leq \int_a^b |y(s)| |y'(s)| ds = \int_a^b \left[|f(s, y(s))|^{-\frac{1}{2}} \right] |y(s)|^{\frac{1}{2}} \left[|f(s, y(s))|^{\frac{1}{2}} |y(s)|^{\frac{1}{2}} |y'(s)| \right] ds.$$

By squaring both sides of (8), applying Schwarz inequality, integrating by parts and using the fact that $y(t)$ is a solution of (C) such that $y(a) = y(b) = 0$ and hypotheses $(H_1) - (H_2)$ we have

$$(9) \quad \begin{aligned} M^4 &\leq \left[\int_a^b \frac{y(s)}{f(s, y(s))} ds \right] \left[\int_a^b (f(s, y(s)) y'(s)) y(s) y'(s) ds \right] = \\ &= \left[\int_a^b \frac{y(s)}{f(s, y(s))} ds \right] \left[- \int_a^b (f(s, y(s)) y'(s)) \frac{y^2(s)}{2} ds \right] = \\ &= \left[\int_a^b \frac{y(s)}{f(s, y(s))} ds \right] \left[\int_a^b g(s, y(s)) \frac{y^2(s)}{2} ds \right] \leq \\ &\leq \left[\int_a^b r(s) ds \right] \left[\frac{1}{2} \int_a^b |g(s, y(s))| |y(s)|^2 ds \right] \leq \end{aligned}$$

$$(9) \quad \leq \frac{1}{2} \left[\int_a^b r(s) ds \right] \left[M^2 \int_a^b w(s, M) ds \right].$$

Now dividing both sides of (9) by M^4 , we get the required inequality in (1).

In order to prove the inequality (2) in Theorem 1, let $y(t)$ be a solution of (D). From the hypotheses, we have

$$M = |y(c)| = \left| \int_a^c y'(s) ds \right| = \left| - \int_c^b y'(s) ds \right|,$$

implying

$$\begin{aligned} 2M &\leq \int_a^b |y'(s)| ds = \\ &= \int_a^b \left[|f(s, y'(s))|^{-\frac{1}{2}} |y'(s)|^{\frac{1}{2}} \right] \left[|f(s, y'(s))|^{\frac{1}{2}} |y'(s)|^{\frac{1}{2}} \right] ds. \end{aligned}$$

The rest of the proof follows by the similar arguments as in the proof of the inequality (1) given above with suitable modifications.

The proof of Theorem 2 follows by closely looking at the proof of Theorem 3 from [1] and the proof of Theorem 2 given in [7] with suitable changes. Here we omit the details.

Remark 1. We note that the results established in this paper can very easily be extended to the general versions of the Eqs. (A) – (D) of the forms:

$$(A') \quad \begin{aligned} &(r(t)|y(t)|^p |y'(t)|^{p-2} y'(t))' + \alpha(t)y'(t) + \beta(t)y(t) + \\ &+ q(t)|y(t)|^{2p-2} y(t) = Q(t), \end{aligned}$$

$$(B') \quad \begin{aligned} &(r(t)|y(t)|^{p-2} |y'(t)|^{p-2} y'(t))' + \alpha(t)y'(t) + \beta(t)y(t) + \\ &+ q(t)|y(t)|^{2p-4} y(t) = Q(t), \end{aligned}$$

$$(C') \quad (f(t, y(t))y'(t))' + \alpha(t)y'(t) + \beta(t)y(t) + g(t, y(t)) = Q(t),$$

$$(D') \quad (f(t, y'(t)))' + \alpha(t)y'(t) + \beta(t)y(t) + g(t, y(t)) = Q(t),$$

where r, p, q, f, g are the same as described in (A) – (D) and $\alpha, \beta, Q: I \rightarrow \mathbb{R}$ are continuous functions and α is continuously differentiable function such that $\frac{1}{2}\alpha'(t) \geq \geq \beta(t)$ for $t \in I$.

For the use of such a condition in the study of certain differential equations, see

[6], [7]. The precise formulations of the results similar to that given in the Theorems 1...4 from [1] and the present two other for the corresponding Eqs. (A') – (D') is very close to that of our results given in this paper with suitable modifications and hence we do not discuss the details.

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ASUPRA UNOR NOI INEGALITĂȚI PRIVIND ZEROURILE SOLUȚIILOR UNOR ECUAȚII DE ORDINUL DOI (II)

(Rezumat)

Se stabilesc noi inegalități ce corelează punctele consecutive în care soluțiile ecuațiilor diferențiale de ordinul doi de tipul (C), (D) prezintă zerouri. Rezultatele sunt stabilite în [1] pentru ecuațiile de tipul (A), (B).

THE FINITE ELEMENT METHODS FOR A STURM-LIOUVILLE PROBLEM

BY

ARIADNA-LUCIA PLETEA

0. Introduction. This paper deals with applying the finite element methods approximation to K -generalized solution for a nonlinear odd degree Sturm-Liouville problem. We have obtained in [4] an unitary theory about applying the finite element methods to a nonlinear equation, generated by nonlinear operator with linear Gâteaux differentiable, K -symmetric and K -positive definite.

1. Statement of the Problem. We consider the problem

$$(1.1) \quad Pu = \sum_{k=0}^{m-1} (-1)^k \frac{d^k}{dx^k} \left[P_{k+1} \left(\frac{d^{k+1}u}{dx^{k+1}} \right) \frac{d^{k+1}u}{dx^{k+1}} \right] = f,$$

$$f \in L^2(a, b),$$

with conditions

$$(1.2) \quad u(a) = u'(a) = \dots = u^{(m-1)}(a) = 0,$$

$$u(b) = u'(b) = \dots = u^{(m-1)}(b) = 0;$$

$P_{k+1} \in C^{k+1}$ class, $k = \overline{0, m-1}$ and P_{k+1} depends only on $d^{k+1}u/dx^{k+1}$.

2. Preliminaries. G. Dinică proved in [2] the existence of K -generalized solution for Eq. (1.1) - (1.2) using variational methods.

Theorem 2.1. We assume that functions P_k satisfy conditions

$$\frac{d}{d\xi} [\xi P_{k+1}(\xi)] \geq c_{k+1} \geq 0, \quad k = \overline{0, m-1}.$$

If there is at least one index k_0 so that

$$(2.1) \quad \frac{d}{d\xi} [\xi P_{k_0+1}(\xi)] \geq c_{k_0+1} > 0, \quad k = \overline{0, m-1},$$

and if the Hilbert space H is $L^2(a, b)$ and

$$Q = \{u \in C^n[a, b] \mid u(a) = u'(a) = \dots = u^{(m-1)}(a) = u^{(m-1)}(b) = 0\},$$

where $n = 2m - 1$ and

$$P: Q \rightarrow L^2(a, b), \quad K = \frac{d}{dx}: Q \rightarrow L^2(a, b),$$

then the operator P is a K -potential operator and $(DP)(u)h$ is linear in h and K -positive definite.

Theorem 2.2. I. If problem (1.1) – (1.2) has a solution, this is the unique solution and minimizes the functional

$$(2.2) \quad (\forall) u \in Q, \quad \mathcal{F}(u) = \sum_{k=0}^{m-1} \int_a^b dx \int_0^{u^{(k+1)}} P_{k+1}(\xi) \xi d\xi - \int_a^b f u' dx.$$

II. Reciprocally, if there is an $u_0 \in Q$ that minimizes the functional (2.2) on Q , then u_0 is the solution of problem (1.1) – (1.2) and, according to the first part of the theorem, the solution is unique.

III. The functional (2.2) is inferiorly bounded on Q .

IV. Any minimized sequence for the functional (2.2) has limit in $L^2(a, b)$ (the existence of K -generalized solution of problem (1.1) – (1.2)).

V. All minimized sequences have the same limit in $L^2(a, b)$ (the uniqueness of K -generalized solution).

We consider H_0 the energetic space of problem (1.1) – (1.2), the completion of Q by virtue of inner product

$$\langle u, v \rangle_0 = \langle (DP)(0)u, Kv \rangle_{L^2(a, b)}, \quad (\forall) u, v \in Q.$$

The corresponding norm $\| \cdot \|_0$ is equivalent to the norm $\| \cdot \|_{H^m(a, b)}$.

Proposition 1.1. If condition (2.1) is true with $c_{k+1} > 0, k = \overline{0, m-1}$, then P is a strong K -monotone operator on H .

Proof. The strong K -monotony results from relations

$$(Pu_1 - Pu_2, K(u_1 - u_2)) = \int_0^1 (DP(tu_1 + (1-t)u_2)(u_1 - u_2), K(u_1 - u_2)) dt \geq$$

$$\geq \frac{\min_{k=\overline{1, m}} c_k}{\max_{k=\overline{1, m}} P_k(0)} \int_0^1 (DP(0)(u_1 - u_2), K(u_1 - u_2)) dt \geq$$

$$\geq \frac{\min_{k=\overline{1, m}} c_k}{\max_{k=\overline{1, m}} P_k(0)} \|u_1 - u_2\|_0^2 = \gamma^2 \|u_1 - u_2\|_0^2$$

and we know from [2] that

$$\|u_1 - u_2\|_0^2 \geq \gamma_1^2 \|u_1 - u_2\|_{L^2(a,b)}^2,$$

$$\|u_1 - u_2\|_0^2 \geq \gamma_2^2 \|K(u_1 - u_2)\|_{L^2(a,b)}^2.$$

Theorem 2.3. *If condition (2.1) is true with $c_{k+1} > 0$, $k = \overline{0, m-1}$, then these sequences are also true.*

I. Any minimized sequence for the functional (2.2) has a limit in H_0 .

II. All minimized sequences for functional (2.2) have the same limit in H_0 .

III. The limit in H_0 for any minimized sequence for functional (2.2) is the K -generalized solution for problem (1.1)–(1.2).

IV. The K -generalized solution of problem (1.1)–(1.2) has generalized derivative to and including m degree.

3. The Finite Element Approximation. On the interval $[a, b]$ we introduce a finite element mesh consisting of nodes

$$a = x_0 < x_1 < \dots < x_N = b$$

the finite element $T_i = (x_i, x_{i+1})$, $i = \overline{0, N-1}$ with the conditions:

$$\bar{T}_i \cap \bar{T}_j = \begin{cases} x_i & \text{if } j = i+1, \\ x_{i+1} & \text{if } j = i-1, \\ \emptyset & \text{if } j \neq i+1, i-1; \end{cases}$$

$$\bigcup_{i=0}^{N-1} T_i = [a, b].$$

We consider the finite element space $H_h \subset L^2(a, b)$ that contains polynomials of degree k , where $k = (\sigma+1)N_i$, σ is the maximum order of derivatives occurring in the defined reference finite element $\hat{T}$, as in [1], and N_i is the number of nodes of the element T_i .

We consider the minimized problem: find a function $u \in H_0$ such that

$$(3.1) \quad \mathcal{F}(u) = \min_{v \in H_0} \mathcal{F}(v)$$

and the discrete problem: find a function $u_h \in H_h$ so that

$$(3.2) \quad \mathcal{F}(u_h) = \min_{v_h \in H_h} \mathcal{F}(v_h).$$

Theorem 3.1. *The minimization problems (3.1) and (3.2) both have one, and only one, solution. Their respective solutions $u \in H_0$ and $u_h \in H_h$ are also the unique solutions of the variational equations:*

$$(3.3) \quad \sum_{k=0}^{m-1} \int_a^b P_{k+1} \left[\frac{d^{k+1}u}{dx^{k+1}} \right] \frac{d^{k+1}u}{dx^{k+1}} \frac{d^{k+1}v}{dx^{k+1}} dx = \int_a^b f \frac{dv}{dx} dx,$$

$$(\forall) v \in H_0,$$

$$(3.4) \quad \sum_{k=0}^{m-1} \int_a^b P_{k+1} \left[\frac{d^{k+1}u_h}{dx^{k+1}} \right] \frac{d^{k+1}u_h}{dx^{k+1}} \frac{d^{k+1}v_h}{dx^{k+1}} dx = \int_a^b f \frac{dv_h}{dx} dx,$$

$$(\forall) v \in H_0.$$

P r o o f. The functional $\mathcal{F}$ is Gâteaux differentiable and

$$(D\mathcal{F})(u)h = \langle Pu - g, Kh \rangle.$$

So, the solutions u and u_h of the minimization problem (3.1) and (3.2) must satisfy relations (3.3) and (3.4) respectively. In view of the strict convexity of the functional $\mathcal{F}$, these relations are also sufficient for the existence of a unique solution.

T h e o r e m 3.2. *The discrete solution u_h is bounded independently of subspace H_h .*

P r o o f. The demonstration is the same as in Lemma 1 from [4] and using

$$\langle Pu_h - Pv, K(u_h - v) \rangle_{L^2(a,b)} \geq |\gamma_1 \gamma_2| \|u_h - v\|_{L^2(a,b)} \|K(u_h - v)\|_{L^2(a,b)}$$

with $v = \theta$, we obtain

$$(3.5) \quad \|u_h\|_{L^2(a,b)} \leq \frac{1}{|\gamma_1 \gamma_2|} \|g - P\theta\|^*.$$

T h e o r e m 3.3. *If u_h is the discrete solution of Eq. (3.4) and if*

$$(3.6) \quad P^k \subset \hat{P} \subset H_0(\hat{T}),$$

$$(3.7) \quad H^m(\hat{T}) \hookrightarrow C^r(\hat{T}),$$

where σ is the greatest derivation degree that appears in definition of $\hat{T}$, the reference finite element, then $(u_h)_{h \in \mathcal{R}}$ is the minimized sequence of functional (2.2).

P r o o f. The space $L^2(a, b)$ being reflexive and (u_h) bounded there exists a subsequence (u_{h_i}) which weakly converges to the same element $u \in L^2(a, b)$. We prove that u is the minimum point of function (2.2) in $L^2(a, b)$, so (u_h) is a minimized sequence.

Let be $\varphi \in \Delta(a, b)$. By definition of the discrete problem we have, in particular

$$(\forall) i \geq 1, \quad \mathcal{F}(u_{h_i}) \leq \mathcal{F}(\Pi_{h_i} \varphi),$$

where Π_{h_i} is the operator of finite element approximation.

Since the functional $\mathcal{F}$ is continuous and convex we have

$$(3.8) \quad \mathcal{F}(u) \leq \lim_{i \rightarrow \infty} \mathcal{F}(u_{h_i}) \leq \lim_{i \rightarrow \infty} \mathcal{F}(\Pi_{h_i} \varphi).$$

Since σ is the maximum order of derivatives occurring interpolation, then, using Theorem 2.10.3 of [3] and from (3.6) and (3.7) we get

$$\|\Pi_{h_i} \varphi - \varphi\|_{L^2(\hat{T})} \leq Ch_{\hat{T}}^m |\varphi|_{m, \hat{T}},$$

so

$$\lim_{i \rightarrow \infty} \|\Pi_{h_i} \varphi - \varphi\|_{L^2(\hat{T})} = 0.$$

This last relation and the continuity of the functional (2.2) imply that

$$\lim_{i \rightarrow \infty} \mathcal{F}(\Pi_{h_i} \varphi) = \mathcal{F}(\varphi)$$

and so, using (3.8) it results that

$$(\forall) \varphi \in \Delta(a, b), \quad \mathcal{F}(u) \leq \mathcal{F}(\varphi).$$

The space $\Delta(a, b)$ being dense in $L^2(a, b)$, it implies that

$$(\forall) v \in L^2(a, b), \quad \mathcal{F}(u) \leq \mathcal{F}(v)$$

and therefore the function u is the unique solution of the minimized problem (3.2), (u_h) is a minimized sequence and u is the K -generalized solution.

From Theorem 2.3 we obtain

Theorem 3.4. *If (u_h) is the sequence of solutions of discrete problem and if u_h weakly converges to the same element u , then $u \in H_0$ and u has generalized derivatives up to the afore mentioned order m inclusively.*

Observation. The same theory can be developed for an even degree Sturm-Liouville equation when $K=I$.

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METODA ELEMENTULUI FINIT APLICATĂ UNEI PROBLEME NELINIARE STURM-LIOUVILLE

(Rezumat)

Se studiază aplicarea metodei elementului finit soluției K -generalizate pentru o problemă neliniară de ordin impar de tip Sturm-Liouville.

$$(2.6) \quad \lim_{n \rightarrow \infty} \|\tilde{u}_n - u\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\tilde{u}_n - u\| = 0$$

Since $\tilde{u}$ is the maximum of $\tilde{u}_n$ over $\tilde{K}$, we have $\tilde{u}_n \leq \tilde{u}$ for all n . From (2.6) and (2.7) we get

$$\|\tilde{u}_n - \tilde{u}\| \leq C\|\tilde{u}_n - u\| \rightarrow 0$$

$$\lim_{n \rightarrow \infty} \|\tilde{u}_n - \tilde{u}\| = 0$$

This last relation and the continuity of the functional (2.5) imply that

$$\lim_{n \rightarrow \infty} J(\tilde{u}_n) = J(\tilde{u})$$

and so, using (2.7) it results that

$$(v) \in \Delta_0 \quad \text{and} \quad J(v) \leq J(\tilde{u})$$

The space Δ_0 being dense in $L^2(\Omega)$, it implies that

$$(v) \in \Delta_0 \quad \text{and} \quad J(v) \leq J(\tilde{u})$$

and therefore the function v is the unique solution of the minimization problem (2.2).

(ii) is a minimum and unique and v is the K -constrained solution.

From Theorem 2.5 we obtain

Theorem 2.4. If (i) is the sequence of solutions of the minimization problem and

(ii) is a minimum and unique and v is the K -constrained solution, then v is the unique

solution of the minimization problem (2.2).

Observation. The same theory can be developed for an even degree Sturm-

Liouville equation when $K = \emptyset$.

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METODA ELEMENTELOR FINITE APLICATĂ ÎN PROBLEME
NELINEARE STURM-LIOUVILLE

Rezumat

Se consideră ecuația de Sturm-Liouville nelineară cu condiții de frontieră omogene. Se demonstrează că soluția este unică și că există o soluție minimă.

FINITE DIFFERENCE APPROXIMATIONS FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

BY

MARIA BECEANU

In this Note we provide estimates of the convergence rate in discrete L^2 norm for finite difference schemes approximating boundary-value problems for degenerate elliptic equations.

In a bounded domain $D \subset \mathbb{R}^n$ with a continuous Lipschitz boundary Γ , degenerate linear elliptic equations of second order are considered.

For the description of the situation we set

$$(1) \quad Au = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left[a_{ij}(x) \frac{\partial u}{\partial x_j} \right] + Bu;$$

B is a non-symmetric perturbation operator while $[a_{ij}(x)]_{i,j=1,n}$ is an Hermitian positive-definite matrix. If $\lambda_i(x)$, $x \in D$ denote its eigenvalues then

$$(2) \quad 0 < \lambda_1(x) \leq \lambda_2(x) < \dots \lambda_n(x) < \infty.$$

The type of degeneration is determined by the behaviour of $\lambda_j(x)$ if $x \rightarrow \partial D$.

There are many possibilities of variation. Of special interest is the following case: if $d(x)$ is the distance of a point $x \in D$ to the boundary and if $(y_1, y_2, \dots, y_n)$ are local coordinates on a neighbourhood V of $z \in \partial D$ such that $V \cap \partial D$ is characterized by $y_n = 0$ and the y_n -direction coincides with the normal direction with respect to ∂D , then this case can be described by

$$(3) \quad Au = - \sum_{i,j=1}^{n-1} \frac{\partial}{\partial y_i} \left[b_{ij}(x) \frac{\partial u}{\partial y_j} \right] - \frac{\partial}{\partial y_n} \left[d(x) \frac{\partial u}{\partial y_n} \right] + \tilde{B} \quad \text{in } V \cap D;$$

$\tilde{B}$ is a perturbation operator, $[b_{ij}(x)]$ is a positive-definite matrix such that

$$(4) \quad \sum_{i,j=1}^{n-1} b_{ij}(x) \xi_i \xi_j \geq c |\xi|^2,$$

where c is a suitable positive constant independent of x . The degeneration takes place

only in normal direction.

As a model problem we consider the following boundary value problem

$$(P) \quad Lu = - \sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i} \left[a_{ij}(x) \frac{\partial u}{\partial x_j} \right] - \frac{\partial u}{\partial x_n} \left[a_{nn}(x) \frac{\partial u}{\partial x_n} \right] + a_0(x)u = f \quad \text{in } D,$$

$$u = 0 \quad \text{on } \Gamma,$$

with the conditions

$$(c_1) \quad a_{ij}(x) \in M(W_2^{s-1}(D)), \quad a_0 \in M(W_2^s(D) \rightarrow W_2^{s-2}(D)), \quad 1 \leq \frac{n}{2} < s \leq 2;$$

$M(V \rightarrow W)$ is the set of the multipliers from V to W . In particular if $V = W$ we set $M(V) = M(V \rightarrow V)$;

$$(c_2) \quad a_{ji}(x) = a_{ij}(x);$$

$$(c_3) \quad a_0(x) \geq c_0 > 0;$$

$$(c_4) \quad \text{the form } \sum_{i,j=1}^{n-1} a_{ij} \zeta_i \zeta_j \text{ is positive-definite for } x \in \bar{D};$$

$(c_5) \quad a_{nn}(x) \geq 0$ in the sense of distributions and vanished on a hypersurface S lying on the $x_n = 0$;

$(c_6) \quad \partial D = \Gamma \cup S$, where Γ is situated in $x_n \geq 0$ and ∂D is a piecewise smooth boundary.

The majority of the estimates are expressed in terms of Sobolev norms. These norms are written in the form

$$\|u\|_{W_2^s(D)}^2 = \sum_{|i| \leq s} \|D^i u\|_{L^2(D)}^2.$$

Use is also made of seminorms in terms of derivatives of one particular order; that is

$$|u|_{W_2^s(D)}^2 = \sum_{|i|=s} \|D^i u\|_{L^2(D)}^2.$$

Assume that the problem (P) has a generalized solution $u \in W_2^s(D)$, $1 \leq n/2 < s \leq 2$. Let $h > 0$. We denote by $R_h^n = R_h \times R_h \times \dots \times R_h$ a uniform square mesh in $\mathbb{R}^n$. Let $\omega \subset R_h^n$ be a bounded connected mesh, $\omega = D \cap R_h^n$. In the set $H(\omega)$ of the functions defined on ω we define the inner product

$$(5) \quad (v, w)_\omega = (v, w)_{L^2(\omega)} = h^n \sum_{x \in \omega} v(x) \cdot w(x).$$

Finite differences are defined in the usual way:

$$(6) \quad v_{x_i} = \frac{v^{+i} - v}{h}, \quad v_{\bar{x}_i} = \frac{v - v^{-i}}{h},$$

where $v^{\pm i} = v^{\pm i}(x) = v(x \pm h e_i)$ and $\{e_1, \dots, e_n\}$ being a basis in $\mathbb{R}^n$.

In the following we will use the averaging operators with the step h

$$(7) \quad T_i^+ f(x) = \int_0^1 f(x + t h e_i) dt = T_i^- f(x + h e_i) = T_i f\left(x + \frac{1}{2} h e_i\right), \quad i = \overline{1, n}.$$

These operators map the partial derivatives onto differences

$$(8) \quad T_i^+(D_i f(x)) = D_i(T_i^+ f(x)) = f_{x_i}(x).$$

The averaging increases the smoothness of the functions; if $f \in W_2^s(D)$ then $T_1 T_2 \dots T_n f \in W_2^{s+1}(D_{h/2})$, where $D_{h/2}$ is the subdomain of D which consists of points whose Euclidean distance from the boundary is exceeding $h/2$.

We approximate the problem (P) on the mesh $\bar{\omega}$ with the following finite difference scheme

$$\mathcal{L}_h v = T_1^2 T_2^2 \dots T_n^2 f \quad \text{in } \omega;$$

$$(P_n) \quad v = 0 \quad \text{on } \Gamma_h = \Gamma \cap \mathbb{R}_h^n,$$

where

$$\mathcal{L}_h v = \frac{1}{2} \left[\sum_{ij=1}^{n-1} (a_{ij} v_{\bar{x}_j})_{x_i} + (a_{ij} v_{x_j})_{\bar{x}_i} + (a_{nn} v_{\bar{x}_n})_{x_n} + (a_{nn} v_{x_n})_{\bar{x}_n} \right] + (T_1^2 \dots T_n^2 a_0) v.$$

Let $D = (0, 1)^n$ and u be the solution of the boundary problem (P) and v be the solution of the difference scheme (P_h) .

For $s > n/2$ $u(x)$ is a continuous function and the error $z = u - v$ is defined at the nodes of the mesh $\bar{\omega}$. Then

$$(*) \quad \mathcal{L}_h z = \sum_{i,j=1}^{n-1} \eta_{ij} z_{\bar{x}_i} + \eta_{nn} z_{\bar{x}_n} + \eta \quad \text{in } \omega,$$

$$z = 0 \quad \text{on } \Gamma_h,$$

where

$$\eta_{ij} = T_i^+ T_1^2 T_2^2 \dots T_{i-1}^2 T_{i+1}^2 \dots T_n^2 \left[a_{ij} \frac{\partial u}{\partial x_j} \right] - \frac{1}{2} (a_{ij} u_{x_j} + a_{ij}^+ u_{\bar{x}_j}^+), \quad i, j = 1, 2, \dots, n-1;$$

$$\eta_{nn} = T_n^+ T_1^2 \dots T_{n-1}^2 \left[a_{nn} \frac{\partial u}{\partial x_n} \right] - \frac{1}{2} (a_{nn} u_{x_n} + a_{nn}^+ u_{\bar{x}_n}^+),$$

$$\eta = (T_1^2 T_2^2 \dots T_n^2 a_0) u - T_1^2 T_2^2 \dots T_n^2 (a_0 u).$$

Remark. The solution z of the problem $(*)$ satisfies the following a priori estimate [5]:

$$(9) \quad \|z\|_{L^2(\omega)} \leq c \left[\sum_{i,j=1}^{n-1} \|\eta_{ij}\|_{L^2(\omega)} + \|\eta_{nn}\|_{L^2(\omega)} + \|\eta\|_{L^2(\omega)} \right].$$

Theorem 1. Let $u \in W_2^s(D)$, $1 \leq n/2 < s \leq 2$. Then the finite scheme (P_h)

converges in the norm of $L^2(\omega)$ and the following estimate holds:

$$(10) \quad \|u - v\|_{L^2(\omega)} \leq ch^{s-1} \|u\|_{W_2^s(D)}.$$

P r o o f. In order to derive estimate (10) we use the bilinear version of the Bramble-Hilbert Lemma [2]. Let $t_i = (\xi_i - x_i)/h$, $i = 1, 2, \dots, n$; $E = \{t = (t_1, t_2, \dots, t_n) / |t_i| < 1, i = 1, \dots, n\}$ and $E_i = \{t = (t_1, t_2, \dots, t_n) / 0 < t_i < 1, |t_k| < 1, k \neq i\}$,

$$a_{ij}(\xi_1, \xi_2, \dots, \xi_n) = \tilde{a}_{ij}(t), \quad i, j = 1, \dots, n-1,$$

$$a_0(\xi_1, \xi_2, \dots, \xi_n) = \tilde{a}_0(t), \quad a_{nn}(\xi_1, \xi_2, \dots, \xi_n) = \tilde{a}_{nn}(t).$$

The right hand-side of (*) can be rewritten in the form

$$\eta_{ij} = \eta_{ij}^1 + \eta_{ij}^2 + \eta_{ij}^3 + \eta_{ij}^4,$$

where

$$\eta_{ij}^1 = T_i^+ T_1^2 \dots T_n^2 \left[a_{ij} \frac{\partial u}{\partial x_j} \right] - (T_i^+ T_1^2 \dots T_n^2 a_{ij}) \left[T_i^+ T_1^2 \dots T_n^2 \frac{\partial u}{\partial x_j} \right], \quad i, j = 1, \dots, n-1;$$

$$\eta_{nn}^1 = T_n^+ T_1^2 \dots T_{n-1}^2 \left[a_{nn} \frac{\partial u}{\partial x_n} \right] - (T_n^+ T_1^2 \dots T_{n-1}^2 a_{nn}) \left[T_n^+ T_1^2 \dots T_{n-1}^2 \frac{\partial u}{\partial x_n} \right];$$

$$\eta_{ij}^2 = \left[T_i^+ T_1^2 \dots T_n^2 a_{ij} - \frac{1}{2} (a_{ij} + a_{ij}^{+i}) \right] \left[T_i^+ T_1^2 \dots T_n^2 \frac{\partial u}{\partial x_j} \right], \quad i, j = 1, 2, \dots, n-1;$$

$$\eta_{nn}^2 = \left[T_n^+ T_1^2 \dots T_{n-1}^2 a_{nn} - \frac{1}{2} (a_{nn} + a_{nn}^{+n}) \right] \left[T_n^+ T_1^2 \dots T_{n-1}^2 \frac{\partial u}{\partial x} \right];$$

$$\eta_{ij}^3 = \frac{1}{2} (a_{ij} + a_{ij}^{+i}) \left[T_i^+ T_1^2 \dots T_n^2 \frac{\partial u}{\partial x_j} - \frac{1}{2} (u_{x_j} + u_{x_j}^{+i}) \right], \quad i, j = 1, 2, \dots, n-1;$$

$$\eta_{nn}^3 = \frac{1}{2} (a_{nn} + a_{nn}^{+n}) \left[T_n^+ T_1^2 \dots T_{n-1}^2 \frac{\partial u}{\partial x_n} - \frac{1}{2} (u_{x_n} + u_{x_n}^{+n}) \right];$$

$$\eta_{ij}^4 = \frac{1}{4} (a_{ij}^{+i} - a_{ij}) (u_{x_j} - u_{x_j}^{+i}), \quad i, j = 1, \dots, n-1;$$

$$\eta_{nn}^4 = \frac{1}{4} (a_{nn}^{+n} - a_{nn}) (u_{x_n} - u_{x_n}^{+n}).$$

For $1 \leq n/2 < s \leq 2$ we take $a_{ij} \in W_p^{s-1}(D)$, $i, j = 1, 2, \dots, n$ and $a_0 = a + \sum_{i=1}^n D_i a_i(x)$, where $a \in L^{2+\varepsilon}(D)$, $\varepsilon > 0$, $a_i \in W_p^{s-1}(D)$, $p > n/(s-1)$, $i = 1, \dots, n$.

We set $\eta = \eta_0 + \eta_1 + \eta_2$, where

$$\eta_0 = (T_1^2 T_2^2 \dots T_n^2 a)u - T_1^2 T_2^2 \dots T_n^2 (au),$$

$$\eta_i = (T_1^2 T_2^2 \dots T_n^2 D_i a_i)u - T_1^2 T_2^2 \dots T_n^2 (u D_i a_i), \quad i = 1, 2, \dots, n.$$

We estimate the term η_{ij}^1 at the node $x \in \omega$,

$$(11) \quad \eta_{ii}^1 = T_i^+ T_1^2 \dots T_n^2 \left[a_{ij} \frac{\partial u}{\partial x_j} \right] - (T_i^+ T_1^2 \dots T_n^2 a_{ij}) \left[T_i^+ T_1^2 \dots T_n^2 \frac{\partial u}{\partial x_j} \right] =$$

$$= \frac{1}{h} \left[\int_{E_i} \prod_{k \neq i} (1 - |t_k|) \tilde{a}_{ij}(t) \frac{\partial \tilde{u}}{\partial t_j} dt - \right.$$

$$\left. - \int_{E_i} \prod_{k \neq i} (1 - |t_k|) \tilde{a}_{ij} dt \int_{E_i} \prod_{k \neq i} (1 - |t_k|) \frac{\partial \tilde{u}}{\partial t_j} dt \right];$$

η_{ij}^1 is a bounded bilinear functional of $(\tilde{a}_{ij}, \tilde{u}) \in W_q^\alpha(E_i) \times W_{2q/(q-2)}^\beta(E_i)$, for $\alpha > 0$, $\beta \geq 1$, $q > 2$. If $\tilde{u}$ is a first-degree polynomial or $\tilde{a}_{ij}$ is a constant then $\eta_{ij}^1 = 0$. By bilinear version of the Bramble-Hilbert Lemma we obtain

$$|\eta_{ij}^1(x)| \leq \frac{c}{h} |\tilde{a}_{ij}|_{W_q^\alpha(E_i)} |\tilde{u}|_{W_{2q/(q-2)}^\beta(E_i)}.$$

Let $J = h^n$ be the Jacobian of the transformation $\zeta = x + th$. Switching back to the original variables we have

$$|\eta_{ij}^1(x)| \leq \frac{c}{h} h^{\alpha+\beta} h^{-n/2} |u|_{W_{2q/(q-2)}^\beta(e_i)} |a_{ij}|_{W_q^\alpha(e_i)},$$

where $e_i = \{\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n); x_i < \zeta_i < x_i + h, |\zeta_k - x_k| < h, k \neq i, k = 1, 2, \dots, n\}$. Summating over nodes of the mesh ω we have

$$\|\eta_{ij}^1(x)\|_{L^2(\omega)} \leq c h^{\alpha+\beta-1} \|a_{ij}\|_{W_q^\alpha(D)} \|u\|_{W_{2q/(q-2)}^\beta(D)}.$$

Setting $\alpha = s - 1$, $\beta = 1$ and $q = p$ from the imbedding Sobolev Theorem we have

$$(12) \quad \|\eta_{ij}^1(x)\| \leq c h^{s-1} \|a_{ij}\|_{W_p^{s-1}(D)} \|u\|_{W_2^s(D)}.$$

By using an analysis similar to that of η_{ij}^2 , for $\eta_{ij}^2, \eta_{ij}^4, \eta_i$ one obtain similar estimates. η_{ij}^3 is a bounded functional of $(a_{ij}, u) \in C(\bar{e}_i) \times W_2^s(e_i)$ which vanishes if u is a first degree polynomial. We obtain for this bilinear functional an estimate of the form (12). η_0 is a bounded bilinear form of $(\tilde{a}, \tilde{u}) \in L^q(e_0) \times W_{2q/(q-2)}^{s-1}(e_0)$ which vanishes if u is a constant. Then we have the following estimate:

$$\|\eta_0\|_{L^2(\omega)} \leq c h^{s-1} \|a\|_{L^{2+\varepsilon}(D)} \|u\|_{W_2^s(D)}, \quad \text{for } 2 < q < \min \left\{ 2 + \varepsilon, \frac{n}{s-1} \right\}.$$

Thus we proved Theorem 1.

The same letter c denotes various constants, not necessarily the same in their various occurrences.

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APROXIMĂRI PRIN DIFERENȚE FINITE PENTRU O CLASĂ DE ECUAȚII ELIPTICE DEGENERATE

(Rezumat)

Se dau estimări ale razei de convergență în norma discretă din L^2 pentru scheme cu diferențe finite care aproximează probleme la limită pentru o clasă de ecuații eliptice degenerate.

MAXIMUM PRINCIPLE FOR CONTROL PROBLEMS GOVERNED BY MULTIVALUED BILINEAR DIFFERENTIAL EQUATIONS

BY

CĂTĂLINA DAVIDEANU

1. Introduction. In this paper a class of control problems is studied and corresponding necessary optimality conditions (maximum principle) are established (Theorem 2). It is a generalization of the particular case which constitutes the subject of article [3]. The existence and the uniqueness of the solution to this type of equations is also presented (Theorem 1).

2. The Control Problem. We consider the control problem: Minimize

$$(P) \quad H(y, v) = \int_0^T \{ [g(t, y(t)) + h(v(t))] \} dt + \varphi_0[y(T)], \quad 0 < T < \infty$$

over the set of all functions $y \in W^{1,2}(0, T, \mathbb{R}^n)$ satisfying the multivalued differential equation

$$(E) \quad y'(t) + (F + \beta)(y(t)) + u(t)F_1(y(t)) \ni f(t) \quad \text{a.e.} \quad t \in]0, T[$$

with the initial condition

$$(IC) \quad y(0) = y_0 \in D(\beta),$$

where $u \in W^{1,2}(0, T)$ is the unique solution of the equation

$$(DE) \quad u'(t) + G(u(t)) = v(t) \quad \text{a.e.} \quad t \in]0, T[$$

with the initial condition

$$(ICDE) \quad u(0) = u_0 \in \mathbb{R} \quad \text{for} \quad v \in L^2(0, T)$$

The above elements have the following properties:

i) $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a locally Lipschitz function and satisfies the condition: there exist the constants $D_1, D_2 \in \mathbb{R}_+$ such that

$$(1) \quad \langle Fy, y \rangle \geq -D_1 \|y\|^2 - D_2 \quad \text{for every } y \in \mathbb{R}^n;$$

ii) $\beta: D(\beta) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a maximal monotone operator with $0 \in \beta(0)$;

iii) $F_1: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a M -Lipschitz function;

iv) $G: \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function which satisfies the condition: there

exist the constants $E_1, E_2 \in \mathbb{R}_+^*$ such that

$$(2) \quad uGu \geq -E_1 \quad |u|^2 - E_2 \quad \text{for every } u \in \mathbb{R};$$

$$v) f \in W^{1,2}(0, T; \mathbb{R}^n);$$

vi) the function $g: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ is measurable in $t, g(t, \cdot) \in L^\infty(0, T)$ and together with $\varphi_0: \mathbb{R}^n \rightarrow \mathbb{R}_+$ satisfies the condition: for every $\delta > 0$ there exists $L_\delta > 0$, independent of t , such that

$$(3) \quad |g(t, y) - g(t, z)| + |\varphi_0(y) - \varphi_0(z)| \leq L_\delta \|y - z\|$$

for all $t \in [0, T]$, $y, z \in \mathbb{R}^n$ with $\|y\| + \|z\| \leq \delta$;

vii) $h: \mathbb{R} \rightarrow \mathbb{R}$ is a convex lower semicontinuous function and there exist $b_1 \in \mathbb{R}_+^*$ and $b_2 \in \mathbb{R}$ such that

$$(4) \quad h(v) \geq b_1 |v|^2 - b_2 \quad \text{for all } v \in \mathbb{R};$$

We can prove that, for every $u_0 \in \mathbb{R}$ and $v \in L^2(0, T)$, the differential equation (DE) with the initial condition (IC) has a unique solution $u \in W^{1,2}(0, T)$.

Definition 1. The function $y: [0, T] \rightarrow \mathbb{R}^n$ is called the strong solution to the equation (E) with the initial condition (IC) if:

a) y is an absolutely continuous function on every compact subset of the interval $]0, T[$,

$$b) y(t) \in D(\beta) \text{ a.e. } t \in]0, T[,$$

$$c) f(t) - y'(t) - F(y(t)) - u(t)F_1(y(t)) \in \beta(y(t)) \text{ a.e. } t \in]0, T[,$$

$$d) y(0) = y_0 \in D(\beta).$$

In this case β is not equal with the subdifferential of a proper, convex and lower semicontinuous function. We cannot apply the following result: if $y \in W^{1,2}(0, T; H)$ so that $y(t) \in D(\varphi(t))$ a.e. $t \in]0, T[$ and there exists $g \in L^2(0, T; H)$ with $g(t) \in \partial\varphi(y(t))$ a.e. $t \in]0, T[$, then the function $t \rightarrow \varphi(y(t))$ is absolutely continuous on $[0, T]$ and we have the equality

$$(5) \quad \frac{d}{dt} \varphi(y(t)) = \langle h, \frac{dy(t)}{dt} \rangle \quad \text{for every } h \in \partial\varphi(y(t)), \text{ a.e. } t \in]0, T[$$

where H is a Hilbert space.

The existence and the uniqueness for the strong solution to the equation (E) with the initial condition (IC) is presented in

Theorem 1. Under the assumptions i) – vii) for every $y_0 \in D(\beta)$ and $(f, u) \in W^{1,2}(0, T; \mathbb{R}^n) \times W^{1,2}(0, T)$ the equation (E) admits a unique solution $y = y(\cdot, y_0, f, u) \in W^{1,\infty}(0, T, \mathbb{R}^n)$.

Moreover y is differentiable from the right at every point of the interval $[0, T]$ and it satisfies the relation

$$(6) \quad \frac{d^+ y}{dt} = (f(t) - F(y(t)) - z(t) - u(t)F_1(y(t)))^0, \quad \forall t \in [0, T],$$

where $z(t) \in \beta(y(t))$ and $(f(t) - F(y(t)) - z(t) - u(t)F_1(y(t)))^0$ represents the element of minimum norm of the set $(f(t) - F(y(t)) - z(t) - u(t)F_1(y(t)))$.

The proof is similar with that presented in [2].

We can prove (see [5]) that the control problem (P) has, at least, one optimal pair.

3. The Maximum Principle is presented in:

Theorem 2. Let $(y^*, v^*) \in W^{1,2}(0, T; \mathbb{R}^n) \times L^2(0, T)$ be an arbitrary optimal pair for the problem (P).

There exists a function $p \in BV([0, T]; \mathbb{R}^n)$ (the dual extremal arc) and a measure $\mu \in ((L^\infty(0, T))^*)^n$ which satisfy the relations

$$(7) \quad p'(t) - \left[\frac{\partial F}{\partial y}(y^*(t)) \right]^* p(t) - u^*(t) \left[\frac{\partial F_1}{\partial y}(y^*(t)) \right]^* p(t) - \\ - \mu \in \partial g(t, y^*(t)) \quad \text{a.e. } t \in]0, T[,$$

$$(8) \quad p(T) + \partial \varphi_0(y^*(T)) \ni 0,$$

$$(9) \quad - \int_t^T U_v^{-1}(t) U_v(\tau) < p(\tau), F_1 y^*(\tau) > d\tau \in \partial h(v^*(t)), \\ \text{a.e. } t \in]0, T[,$$

where U_v satisfies the relations

$$(10) \quad U'_v(t) + G'(u(v)) U_v(t) = 0,$$

$$(11) \quad U_v(0) = 1.$$

Proof The idea of the proof consists in

a) approximating the control problem (P) by a family of smooth problems (P^ε) , $\varepsilon > 0$ (for which the necessary optimal conditions are immediate);

b) passing to the limit in the approximate optimality conditions ([1]).

For every $\varepsilon > 0$ we construct the problem: Minimize

$$(P^\varepsilon) \quad H^\varepsilon(y, v) = \int_0^T (g^\varepsilon(t, y(t)) + h(v(t)) + \frac{1}{2} |v(t) - v^*(t)|^2) dt + \varphi_0^\varepsilon(y(T)), \quad 0 < T < \infty$$

over the set of all functions $y \in W^{1,2}(0, T; \mathbb{R}^n)$ which satisfy the approximate differential equation.

$$(E^\varepsilon) \quad y'(t) + (F^\varepsilon + \beta^\varepsilon)(y(t)) + u(t) F_1^\varepsilon(y(t)) = f(t) \quad \text{a.e. } t \in]0, T[,$$

with the initial condition (IC), where $u \in W^{1,2}(0, T)$ is the unique solution to the equation

$$(DE^\varepsilon) \quad u'(t) + G^\varepsilon(u(t)) = v(t) \quad \text{a.e. } t \in]0, T[,$$

with the initial condition $(IC)_1$, for $v \in L^2(0, T)$.

The elements which appear in (E^ε) and (DE^ε) are presented in [4].

Lemma 1. The following convergences are true:

i) $v^\varepsilon \rightarrow v^*$ weakly in $L^2(0, T)$;

ii) $u^\varepsilon \rightarrow u^* = u(\cdot, u_0, v^*)$ strongly in $C(0, T)$;

- iii) weakly in $W^{1,2}(0, T)$;
- vi) $y^\varepsilon \rightarrow y^* = y(\cdot, y_0, u^*)$ strongly in $C([0, T]; \mathbb{R}^n)$;
- v) for $\varepsilon \rightarrow 0$ weakly in $W^{1,2}(0, T; \mathbb{R}^n)$

For the prove see [1].

If $(y^\varepsilon, v^\varepsilon)$ is an optimal pair for the problem (P^ε) then there exists the inequality

$$(12) \quad H^\varepsilon(y^\varepsilon, v^\varepsilon) \leq H^\varepsilon(y^\varepsilon_\lambda, v^\varepsilon + \lambda \bar{v})$$

for every $\varepsilon > 0$ and $\bar{v} \in L^2(0, T)$.

We have denoted by $y^\varepsilon_\lambda(t) = y(t, y_0, u^\varepsilon_\lambda)$ the unique solution to the differential equation (DE^ε) corresponding to the unique solution $u^\varepsilon(t) = u(t, u_0, v^\varepsilon + \lambda \bar{v})$ to the equation (E^ε) .

The inequality (12) can be written as

$$(12') \quad \int_0^T \left(\langle \nabla g^\varepsilon(t, y^\varepsilon(t)), \bar{y}^\varepsilon(t) \rangle + h'(v^\varepsilon, \bar{v}) + \langle v^\varepsilon - v^*, \bar{v} \rangle \right) dt + \\ + \langle \nabla \varphi_0(y^\varepsilon(T)), \bar{y}^\varepsilon(T) \rangle \geq 0 \quad \text{for all } \bar{v} \in L^2(0, T).$$

Let $p^\varepsilon \in BV([0, T]; \mathbb{R}^n)$ be the unique solution to the equation

$$(13) \quad p^{\varepsilon'}(t) - (\nabla F^\varepsilon(y^\varepsilon) + \nabla \beta^\varepsilon(y^\varepsilon))^* p^\varepsilon(t) - u^\varepsilon(t) (\nabla F^\varepsilon_1(y^\varepsilon))^* p^\varepsilon(t) = \\ = \nabla g^\varepsilon(t, y^\varepsilon) \quad \text{a.e. } t \in]0, T[,$$

with the final condition

$$(13') \quad p^\varepsilon(T) + \nabla \varphi_0(y^\varepsilon(T)) = 0,$$

whose existence is established in [5].

We consider the system:

$$(14) \quad \frac{d\bar{y}^\varepsilon}{dt}(t) + (\nabla F^\varepsilon(y^\varepsilon) + \nabla \beta^\varepsilon(y^\varepsilon)) \bar{y}^\varepsilon(t) + u^\varepsilon \nabla F^\varepsilon_1(y^\varepsilon) \bar{y}^\varepsilon(t) + \\ + \bar{u}^\varepsilon(t) F^\varepsilon_1(y^\varepsilon) = 0 \quad \text{a.e. } t \in]0, T[,$$

$$(15) \quad \frac{dp^\varepsilon}{dt}(t) - (\nabla F^\varepsilon(y^\varepsilon) + \nabla \beta^\varepsilon(y^\varepsilon))^* p^\varepsilon(t) - u^\varepsilon (\nabla F^\varepsilon_1(y^\varepsilon))^* p^\varepsilon(t) = \\ = \nabla g^\varepsilon(t, y^\varepsilon).$$

Taking the scalar product of the first equation with p^ε and the scalar product of the second equation with y^ε and adding we get

$$(16) \quad \frac{d}{dt} \langle p^\varepsilon, \bar{y}^\varepsilon \rangle = \langle \nabla g^\varepsilon(t, y^\varepsilon), \bar{y}^\varepsilon \rangle - \langle p^\varepsilon, \bar{u}^\varepsilon F^\varepsilon_1(y^\varepsilon) \rangle.$$

Integrating relation (16) and using the initial condition $\bar{y}^\varepsilon(0) = 0$ and the final condition $p^\varepsilon(T) = -\nabla \varphi_0(y^\varepsilon(T))$ we have

$$(16') \quad \begin{aligned} & \langle p^e(T), \bar{y}^e(T) \rangle - \langle p^e(0), \bar{y}^e(0) \rangle = \int_0^T \langle \nabla g^e(t, y^e), \bar{y}^e \rangle dt - \\ & - \int_0^T \bar{u}^e \langle p^e, F_1^e(y^e) \rangle dt, \end{aligned}$$

$$\text{or} \quad -\langle \nabla \varphi_0^e(y^e(T), \bar{y}^e(T)) \rangle = \int_0^T \langle \nabla g^e(t, y^e), \bar{y}^e \rangle dt - \int_0^T \bar{u}^e \langle p^e, F_1^e(y^e) \rangle dt.$$

Thus

$$(17) \quad \begin{aligned} & \int_0^T \langle \nabla g^e(t, y^e), \bar{y}^e \rangle dt + \langle \nabla \varphi_0^e(y^e(T), \bar{y}^e(T)) \rangle = \\ & = \int_0^T \bar{u}^e \langle p^e, F_1^e(y^e) \rangle dt. \end{aligned}$$

Now, the relation (12') becomes

$$(12'') \quad \int_0^T (\bar{u}^e \langle p^e, F_1^e(y^e) \rangle + h'(v^e, \bar{v}) + \langle v^e - v^*, \bar{v} \rangle) dt \geq 0$$

for every $\bar{v} \in L^2(0, T)$.

The Fréchet differential $\bar{u}^e$ of the application $v^e \rightarrow u^e(v^e)$ satisfies the relations:

$$(18) \quad \begin{aligned} & \bar{u}^e(t) + G^{e'}(u^e(t)) \bar{u}^e(t) = \bar{v}(t) \quad \text{a.e. } t \in]0, T[, \\ & \bar{u}^e(0) = 0, \end{aligned}$$

so that

$$(19) \quad \bar{u}^e(t) = U_v^e(t) \int_0^t (U_v^e(s))^{-1} \bar{v}(s) ds,$$

where

$$\begin{aligned} & (U_v^e(t))' + G^{e'}(u^e(t)) U_v^e(t) = 0, \\ & U_v^e(0) = 1. \end{aligned}$$

Replacing $\bar{u}^e$ given by (19) in (12'') we obtain

$$(20) \quad \begin{aligned} & \int_0^T \left\langle \left(U_v^e(s) \right)^{-1} \int_s^t p^e(t) U_v^e(t) F_1^e(y^e) dt, \bar{v} \right\rangle dt + \\ & + \int_0^T h'(v^e, \bar{v}) dt + \int_0^T \langle v^e - v^*, \bar{v} \rangle dt \geq 0 \end{aligned}$$

for every $\bar{v} \in L^2(0, T)$ which implies

$$(20') \quad - \int_0^T \left(U_v^\varepsilon(t) \right)^{-1} U_v^\varepsilon(\tau) \langle p^\varepsilon(\tau), F_1^\varepsilon(y^\varepsilon(\tau)) \rangle d\tau \in \partial h(v^\varepsilon(t)) + v^\varepsilon(\tau) - v^*(\tau) \quad \text{a.e. } t \in]0, T[.$$

The relations (13), (13') and (20') represent the necessary optimality conditions for the approximate control problem (P^ε) .

When ε approaches 0 these equations become the relations (7), (8), (9).

Remark. The particular case when $\beta = \partial I_K$ where ∂I_K is the subdifferential of the indicator function of the convex and closed set

$$K = \{z \in \mathbb{R}^n; 0 \leq z \leq a_i, i = \overline{1, n}, a_i \in \mathbb{R}_+^*, i = \overline{1, n}\}$$

has been studied in [3].

The control problem becomes: Minimize

$$(P_K) \quad \int_0^T (g(t, y(t)) + h(u(t)) dt + \varphi_0(y(T)), \quad 0 < T < \infty$$

over the set of all functions $(y, u) \in W^{1,2}(0, T; \mathbb{R}^n) \times L^2(0, T)$ which satisfy the variational inequality

$$(V_K) \quad y'(t) + (F + \partial I_K)(y(t)) + u(t) F_1(y(t)) \ni f(t) \quad \text{a.e. } t \in]0, T[,$$

with the initial condition

$$y(0) = y_0 \in K.$$

The maximum principle for this particular problem (P_K) has been established in [3].

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PRINCIPIUL DE MAXIM PENTRU PROBLEME DE CONTROL GUVERNATE
DE ECUAȚII DIFERENȚIALE MULTIVOCE BILINIARE

(Rezumat)

Se prezintă teorema de existență și unicitate pentru soluția unei astfel de ecuații diferențiale multivoce. De asemenea, se stabilesc condițiile necesare de optimalitate (principiul de maxim).

Problema de control optimal studiată constituie o generalizare a celei considerate în articolul [3].

PRINCIPIUL DE MAXIM PENTRU PROBLEME DE CONTROL GUVERNATE
DE ECUAȚII DIFERENȚIALE MULTIVOCE BILINIARE

(Rezumat)

Se prezintă teorema de existență și unicitate pentru sistemele analitice de ecuații diferențiale multivoce. În asemenea condiții necesare de optimizabilitate (principiul de maxim) problema de control optim analitic constă în generalizarea a celui considerat în acest articol.

(1)

ESTIMATION AND PROPERTIES OF AVERAGE ENERGIES BY BAND, CENTROIDS AND SPURIOUS POINTS FOR SPANISH VOCALIC UTTERANCES IN A SYSTEM OF AUTOMATIC SPEECH RECOGNITION

BY

P. M. GADEA and GUSTAVO SANTOS-GARCÍA

1. Introduction. The practical implementation of the different methods or systems of Automatic Speech Recognition gives rise to many problems. In the present paper we treat in some detail three of them: the determination of the average energies by band (§ 2), that of centroids (§ 3) and an algorithm of recognition and dropping of spurious points (§ 4). These procedures are useful in a number of determinations of characteristics in questions of Automatic Speech Recognition. We consider here the specific case of the five Spanish vowels. For more details see [1], [2], [4]. For the sake of brevity we present here only the graphics corresponding to a few cases of the Spanish vowels. The whole of the cases can be viewed in [4].

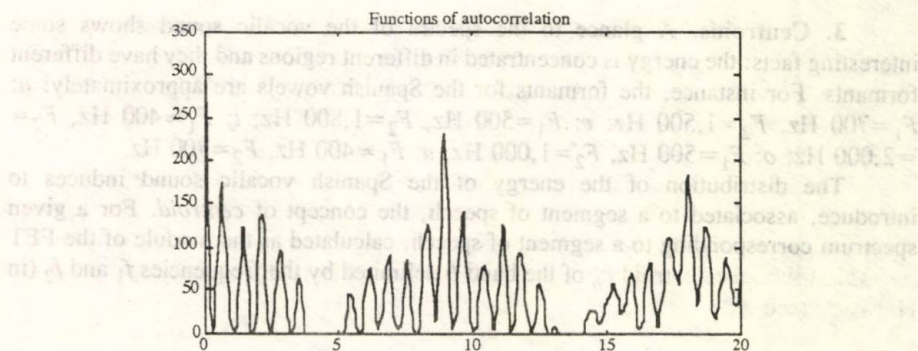


Fig. 1 — Functions of autocorrelation. Determination of the fundamental frequency.

2. Average Energies by Band. As is well-known, when the spectral represent-

ation of the speech signal is given in terms of the Fourier transform, we have a speech recognition procedure by means of the calculation of the normalized energies by band in the successive segments of speech.

In Fig. 1 we show the fast Fourier transform (FFT) of 1024 points which we have calculated for segments of 10 msec in the pronunciation of each of the five Spanish vowels. As one may see, the distribution of the energy is very varied. We have divided the interval of useful frequencies — i.e., [200 Hz, 4000 Hz] — into 15 intervals of the same length. For each of the bands we take the quotient of its energy by the total amount of energy in the given interval. Since the distribution of energy is different for each sound, one can use these parameters of normalized energy by band as input data in the systems of recognition (pattern recognition, Neural Networks, etc.). The use of these parameters is especially interesting due to three properties:

- i) the stability of the parameters of a given type of vocalic sound;
- ii) the values of the parameters corresponding to different vowels are distant;
- iii) they can be implemented in real time, since we use the FTT.

In the Figs. 2 we show the normalized energies by band for the Spanish vowels. We have used a bank of data with 50 speakers and three realizations for each of them. We thus have 150 values of the energy by vowel. In each of the figures, we represent the normalized energies by band for each band and for all the realizations of the corresponding vowel. We first see in the Figs. 2, that for each vowel and each band, the values of the normalized energies are very near, except for a few spurious points. We also see the differences among the five vowels. The normalized energies by band may so be useful as input parameters in the techniques of pattern recognition, statistic classifiers or Neural Networks.

3. Centroids. A glance to the spectra of the vocalic sound shows some interesting facts: the energy is concentrated in different regions and they have different formants. For instance, the formants for the Spanish vowels are approximately: *a*: $F_1=700$ Hz, $F_2=1,500$ Hz; *e*: $F_1=500$ Hz, $F_2=1,800$ Hz; *i*: $F_1 \approx 400$ Hz, $F_2 = 2,000$ Hz; *o*: $F_1=500$ Hz, $F_2=1,000$ Hz; *u*: $F_1 \approx 400$ Hz, $F_2=700$ Hz.

The distribution of the energy of the Spanish vocalic sound induces to introduce, associated to a segment of speech, the concept of *centroid*. For a given spectrum corresponding to a segment of speech, calculated as the module of the FFT of 1024 points, the centroid c_b of the band b delimited by the frequencies f_1 and f_2 (in Hz) is defined as

$$(3.1) \quad c_b = \frac{\sum_{i=n_1}^{n_2} f(i) a(i)}{\sum_{i=1}^{n_2} a(i)},$$

where n_1 and n_2 are the indices of the FFT [4] on the frequency axis corresponding to the frequencies f_1 and f_2 respectively, $f(i)$ — the frequency in Hz corresponding to the i -th frequency index, and $a(i)$ — the normalized amplitude corresponding to the frequency $f(i)$ (in Hz) or to the frequency index i .

An alternative way of defining the centroid c_b corresponding to the band b , delimited by the frequencies f_1 and f_2 (in Hz) is as the frequency in which the energies in the sub-bands delimited by the frequencies f_1 and c_b (in Hz) and by the frequencies c_b and f_2 (in Hz) respectively coincide.

The use of bands and centroids permits to obtain a series of parameters distinguishing the sounds which can be used in an efficient way in the algorithms of pattern recognition, as in the case of the normalized energies by band.

In the present paper we consider two bands of frequency: 0...1,925 Hz and 1,925...3,850 Hz, so permitting to calculate the centroids of each band according the previous definition. Moreover, we can determine other parameters which are significant in the differentiation of sounds. First, we define the *relation r of normalized energy between bands*

$$(3.2) \quad r = \frac{e_{b_1}}{e_{b_2}},$$

where e_{b_1} and e_{b_2} are the energies of the first band b_1 and the second band b_2 , calculated as

$$(3.3) \quad e_{b_i} = \sum_{i=n_{i1}}^{n_{i2}} a(i).$$

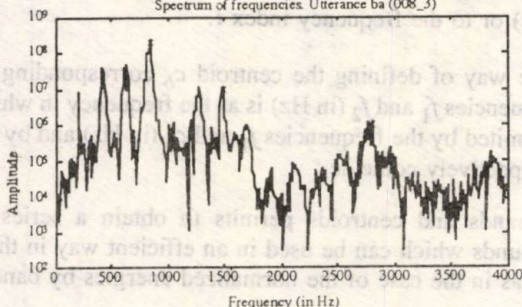
Another distinguishing parameter is the distance d between the centroids of the bands, i.e., $d = c_{b_2} - c_{b_1}$. One can see experimentally that d takes very different values for the considered five vocalic sounds. We present in the following figures the graphic results obtained for the centroids and the relation of energy between bands.

The respective values for the first and the second centroid of the five vowels is: vowel a : 700 Hz and 2,800 Hz; vowel e : 450 Hz and 2,600 Hz; vowel i : 300 Hz and 2,700 Hz; vowel o : 500 Hz and 3,000 Hz; vowel u : 350 Hz and 2,600 Hz.

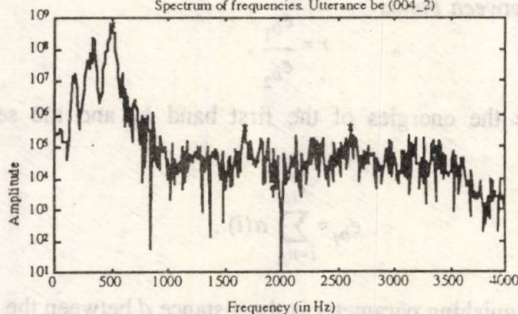
We see remarkable differences and also that there exists a narrow relation of these centroids with the known values of the first two formants of the above vowels. A glance to the graphics shows other interesting characteristics that we omit for the sake of brevity.

For the calculation and storage of the values of the centroids and the relation of energies, for the files of sounds of our database, we have written the following program, developed in MatLab:

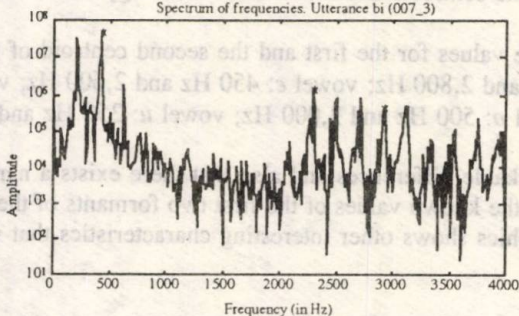
Spectrum of frequencies. Utterance ba (008_3)



Spectrum of frequencies. Utterance be (004_2)



Spectrum of frequencies. Utterance bi (007_3)



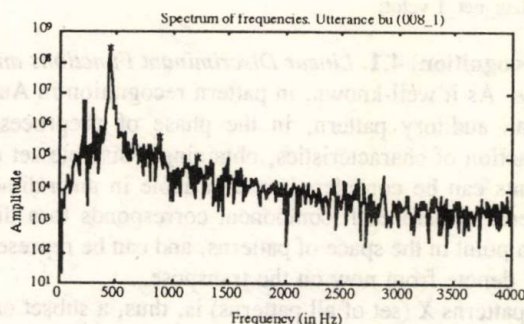
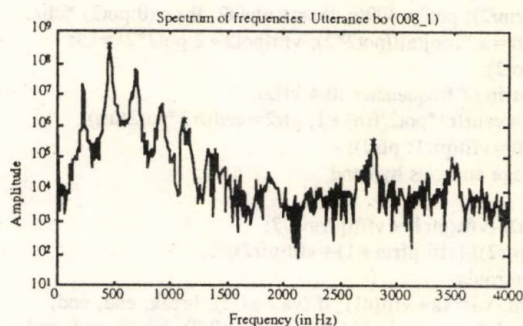


Fig. 2 — Representation of the spectrum and the first three formant frequencies (marked with an asterisk) for the five Spanish vowels and a single speaker.

centroid.m

function name=centroid (name)

% Calculation of centroids in the spectrum of the trace of the vowels

%nvow=name(6); % in the case of a file of type al011a

nvow=name(2); % in the case of a file of type ba001_1

% We assign to each utterance a number of vowel or consonant

nv=numbvw (nvow);ns=nv;

load data_net_1

% It loads the data of the file of data of sounds

eval (['load ',name]);

% We select an interval of the trace of sound

```

beginning=3000; a1=beginning+1; a2=beginning+2000;
eval(['utter=',name,'(',num2str(a1),',',num2str(a2),')']);
tam=length(utter);
% It computes the fast Fourier transform
frm=22000; frn=ceil(frm/2); pot2=4096; dhz=frn/pot2; fhz=(0:pot2)*dhz;
a=fft(utter, pot2*2); vfft=a.*conj(a)/(pot2*2); vfft(pot2+2:pot2*2)=[];
vfft(2:pot2)=2*vfft(2:pot2);
% We consider only a strip of frequencies (0-4 kHz),
fr1=0; fr2=3850; pfr1=ceil(fr1*pot2/frn)+1; pfr2=ceil(fr2*pot2/frn);
fhz=fhz(pfr1:pfr2); vfft=vfft(pfr1:pfr2);
% It calculates the average energies by band
pfrm=fix(pfr2/2);
a1=sum(vfft(pfr1:pfrm))-(vfft(pfr1)+vfft(pfrm))/2;
a2=sum(vfft(pfrm+1:pfr2))-(vfft(pfrm+1)+vfft(pfr2))/2;
% Calculation of the centroids
va=0; for c1=pfr1:pfrm, va=va+vfft(c1); if (va>a1/2), break, end, end,
va=0; for c2=pfrm+1:pfr2, va=va+vfft(c2); if (va>a2/2), break, end, end,
a1=a1*dhz; a2=a2*dhz; c1=c1/1000*dhz; c2=c2/1000*dhz; % we convert to Hz
rate=a2/a1; vc=[ns c1 c2 rate];
% It stores the parameters in the corresponding file
vctot=[vctot;vc]; save data_net_1 vctot;

```

4. Pattern Recognition. 4.1. Linear Discriminant Functions and the Classifier of Minimum Distance. As it well-known, in pattern recognition in Automatic Speech Recognition, from an auditory pattern, in the phase of preprocess of data, one proceeds to the extraction of characteristics, obtaining a discrete set of n data. Each of those n components can be considered as a variable in an n -dimensional space, called space of patterns, where each component corresponds to a dimension. Each pattern x appears as a point in the space of patterns, and can be represented as $x = {}^t(x_1, x_2, \dots, x_n)$, where ${}^t(\)$ denote from now on the transpose.

The space of patterns X (set of all patterns) is, thus, a subset of $\mathbb{R}^n$, supposing the data take real values, as in the present case.

The non-parametric algorithms of pattern recognition are used, as is well-known, when no assumptions can be made on the parameters of characteristics. The patterns belonging to different classes assemble in different regions in the space of patterns, which can be easily separated by the hypersurfaces of decision. These are usually hyperplanes represented by $w_1x_1 + w_2x_2 + \dots + w_nx_n + w_{n+1} = 0$ or in matrix form by $wx = 0$, where $w = {}^t(w_1, w_2, \dots, w_n, w_{n+1})$ and $x = {}^t(x_1, x_2, \dots, x_n, 1)$ correspond to the vector of weights and to the vector of patterns, respectively. A discriminant function is a function $d(x)$ defining a hypersurface of decision. Let $d_k(x)$ and $d_j(x)$ be the values of the discriminant function for patterns x of the classes k and j , respectively. The equation $d(x) = d_k(x) - d_j(x) = 0$ defines the hypersurface separating the classes k and j . The discriminant function satisfies $d_k(x) > d_j(x)$, $x \in \omega_k$, $j \neq k$, $j = 1, 2, \dots, m$. If one has m classes, the number of hypersurfaces of decision we can take is $m(m-1)/2$. Actually, it suffices with $m-1$ to separate the m classes.

A linear discriminant function has the form $d_k = w_{k1}x_1 + w_{k2}x_2 + \dots + w_{kn}x_n + w_{k, n+1} = 0$, or, in matrix form, $d_k(x) = {}^t w_k x$, where $w_k = {}^t(w_{k1}, w_{k2}, \dots, w_{kn}, w_{k, n+1})$.

$w_{k,n+1}$) and $x = (x_1, x_2, \dots, x_n, 1)$. This function permits us to define the *classifier of minimum distance*. The rule of decision used in this method is

$$(4.1) \quad x \in \omega_j \quad \text{if} \quad \delta(x, z_j) = \min_k \delta(x, z_k), \quad k = 1, 2, \dots, m,$$

where $\delta(\cdot)$ is the distance, usually the Euclidean distance, of an unknown pattern x to the average prototype z_k — center of the class for the class w_k . If one considers the Euclidean distance, the rule of decision expressed in (4.1) is equivalent to seeking $\max_k (x z_k - \frac{1}{2} z_k z_k)$, $k = 1, 2, \dots, m$. The discriminant function used in the classifier can be expressed as

$$(4.2) \quad d_k(x) = x z_k - \frac{1}{2} z_k z_k - \frac{1}{2} |z_k|^2 = x w,$$

where x is the vector of the pattern and $w = (z_k^1, z_k^2, \dots, z_k^n, -\frac{1}{2} |z_k|^2)$. The hypersurface of decision between two classes ω_i and ω_j remains expressed as

$$(4.3) \quad d(x) = x(z_i - z_j) - \frac{1}{2}(z_i z_i - z_j z_j) = 0,$$

which corresponds to the hyperplane of points equidistant of the centers z_i and z_j of the corresponding classes.

4.2. Algorithm for Recognition of Spurious Points. This algorithm of formation of clusters is a variant of the algorithm of minimum distance, in which, when determining the centers, a proportion of points considered as spurious or end cases of a class is dropped. We have seen examples of such points in the graphics of the previous §§. The method of elimination of spurious points consists in the following steps:

1. Fix the total number k of classes to be considered in the total space of patterns and the proportion r of patterns to be dropped in each process.

2. Drop from the space of patterns a proportion $r\%$ of patterns, which will be considered spurious. The remaining points constitute the generic cluster.

Take at random a point of the generic cluster. Consider the set of points of the generic cluster whose distance to the given point is less than a fixed distance. Calculate the center of that cluster and form a new cluster, now constituted by the points whose distance to that center is less than the previous given distance. Repeat this process q times. The last cluster — the q -th one — is dropped from the generic one.

3. Repeat step 2 a total of k times, verifying that the number of points of the "residual" cluster is small.

4. Debug the clusters dropping an $r\%$ of points in each cluster.

The implementation in MatLab of the main program with which we have developed this algorithm is:

clusters.m

function nam=clusters(nam)

```

% Model of "several clusters and rest".
% Inicialization of files and variables
% Load of the file with the set of points where to determine the clusters
eval(['load ',nam]);
N=vlpc; % We load the set of points in the matrix N
clear vlpc;
% Step 1. Dropping of spurious points
[n, m]=size(N);
ne=10; % Number of points to be dropped
Pm=mean(N);
for j=1:n,dist0(j)=norm(N(j,3:m)-Pm(3:m));end, % initial position
for i=1:ne,
Pm=mean(N);for j=1:n+1-i,diste(j)=norm(N(j,3:m)-Pm(3:m));end,
[vmax, jmax]=max(diste);clear diste;N(jmax,:)=[];
end
Pm=mean(N);
for j=1:n-ne,distf(j)=norm(N(j,3:m)-Pm(3:m));end, % final situation
dM=max(distf);dm=min(distf); % maximum and minimum distance to the center of mass
clg;subplot(221)
plot(dist0);title(['distances to the (initial) center',nam]);
plot(distf);title(['distances to the center (without spurious)',nam]);
% Step 2. Determination of a cluster.
[n,m]=size(N);
P=N(1,:); % a point of the cluster
rm1=(dM-dm)/2; % value of the radius of the ball to be considered
n1=0; % counter of points of the cluster
for j=n:-1:1,
distanc=norm(N(j,3:m)-P(3:m));
if distanc < rm1,n1=n1+1;N1(n1,1:m)=N(j,:);N(j,:)=[];end,
end
Pm1=mean(N1);for j=1:n1,distN1(j)=norm(N1(j,3:m)-Pm1(3:m));end,
plot(distN1);title(['distances to the center (cluster 1)',nam]);
% Step 3. Determination of other cluster
[n,m]=size(N); % the farthest point of the previous cluster
for j=1:n,distU(j)=norm(N(j,3:m)-Pm1(3:m));end,
[vmax, jmax]=max(distU);P=N(jmax,:);
rm2=min(rm1,vmax-rm1); % value of the radius of the ball to be considered
n2=0; % counter of points of the cluster
for j=n:-1:1,
distanc=norm(N(j,3:m)-P(3:m));
if distanc < rm2,n2=n2+1;N2(n2,1:m)=N(j,:);N(j,:)=[];end,
end
Pm2=mean(N2);for j=1:n2,distN2(j)=norm(N2(j,3:m)-Pm2(3:m));end,
plot(distN2);title(['distances to the center (cluster 2)',nam]);
% Step 4. cluster 'Rest'.
[n,m]=size(N);
Pm=mean(N);for j=1:n,distN(j)=norm(N(j,3:m)-Pm(3:m));end,
rm=max(distN);
plot(distN);title([nam,' dist. center (cluster rest) n=',num2str(n)]);
% Step 5. Representation and filing of results.
nam(1:4)='fvar';

```



```

listvar=' ne dist0 distf dM dm rm1 n1 N1
Pm1 distN1 rm2 n2 N2 Pm2 distN2 rm n N Pm distN;';
eval(['save ',nam,listvar]);
for j=1:n2,dist12(j)=norm(N2(j,3:m)-Pm1(3:m));end,
for j=1:n,dist10(j)=norm(N(j,3:m)-Pm1(3:m));end,
plot(dist12);title(['distance of center 1 to cluster 2 -',nam]);
plot(dist10);title(['distance of center 1 to cluster rest -',nam]);
for j=1:n1,dist21(j)=norm(N1(j,3:m)-Pm2(3:m));end,
for j=1:n,dist20(j)=norm(N(j,3:m)-Pm2(3:m));end,
plot(dist21);title(['distance of center 2 to cluster 1 -',nam]);
plot(dist20);title(['distance of center 2 to cluster rest -',nam]);

```

Appendix. Used Hardware and Software

Hardware. 1) Computer Apple Macintosh, with a processor 68030 to 16 MHz and mathematical coprocessor 68882 to 16 MHz. Hard disk: 80 Mb. RAM memory: 4 Mb. For the storage of the database of sounds, we have disposed of a removable hard disk. 2) Digitizer of sound MacRecorder. It permits to record the sounds to different frequencies of sampling (5, 7, 11 and 22 kHz). The quantization of the digitizer is of 8 bits, having so 256 quantization levels. Mono and stereo.

Software. SoundEdit, editor of sound managing the digitizer MacRecorder, generating files with format 'snd', which are then translated to numeric format by using our own techniques, to be used with the mathematical environment MatLab, with which we have developed the treatment of the digital waveform and the proposed algorithms of recognition. We have selected the system of development MPW Pascal 2.0.2. The used environment of programming offers the languages Pascal, C and assembler.

Notation and methodology. The samples of voice have been obtained from a set of 50 speakers, half male and half female, with ages from 20 to 35. We have so reliable results from a statistical point of view. Each speaker has pronounced 3 times each of the 5 Spanish vowels. The frequency of sampling has been 22 kHz. As the sampling theorem establishes the Nyquist frequency being half the frequency of sampling, we have avoided the problem of "aliasing". The conversion from analog to digital signal has been taken with a resolution of 8 bits. The frequency of sampling and the resolution of digitization suppose a relation of requirement of storage by time of 21.48 Kbytes/sec.

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ESTIMAREA ȘI PROPRIETĂȚILE ENERGIEI MEDII PE BENZI,
CENTROIDELOR ȘI PUNCTELOR ALTERATE PENTRU
VOCALELE SPANIOLE ÎNTR-UN SISTEM DE
RECUNOAȘTERE AUTOMATĂ A VORBIRII

(Rezumat)

În cadrul aplicării diverselor metode de recunoaștere automată a vorbirii se studiază aspecte energetice și algoritmi care privesc direct cele cinci vocale ce intervin în limba spaniolă.

LES TYPES DE CIRCULATIONS ADMISSIBLES DANS UN GRAPHE ESCALIER NON ORIENTÉ (II)

PAR

VIOREL GHIUȘ

Nous étudions le problème de la transformation d'une circulation admissible dans un graphe escalier G^f , où f est une orientation $\bar{b}/c' \bar{c}/b$ ou $\bar{b}'/c \bar{c}/b'$ (équivalentes) [5] dans des circulations admissibles plus simples, ou circulations admissibles dans des graphes avec une structure plus simple.

Théorème 1. *Pour toute circulation admissible de type $\bar{b}/c' \bar{c}/b$ dans un graphe escalier (Fig. 1) [5], il existent deux circulations standards admissibles en des graphes escaliers [3].*

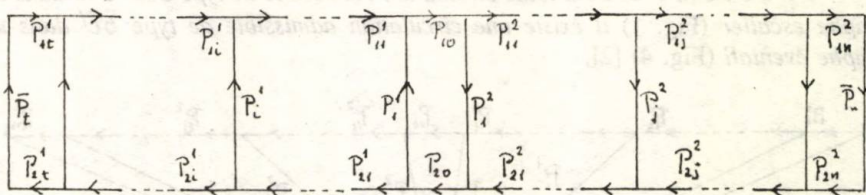


Fig. 1

Démonstration. Soit un graphe escalier non orienté G avec une circulation admissible (f, F) (Fig. 1), où f est une orientation $\bar{b}/c \bar{c}/b$ ou $\bar{b}'/c \bar{c}/b'$ de G et F une circulation dans G^f (le graphe orienté attaché); les flèches du graphe indiquent l'orientation $\bar{b}/c' \bar{c}/b$.

Nous détachons les arêtes P_{10} , respectivement P_{20} , dans le sommet x , respectivement y (Fig. 2), et nous collons les extrémités du sommet y , respectivement x ; le nouveau graphe G' (Fig. 3) peut être écrit comme une somme directe de deux graphes disjointes-sommets

$$(1) \quad G' = G_1 + G_2.$$

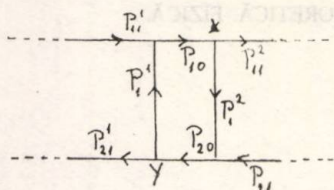


Fig. 2

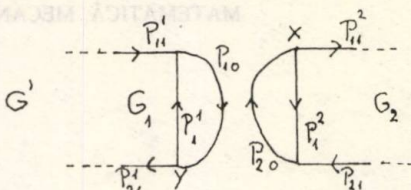


Fig. 3

Les équations de conservation sont évidemment satisfaites, parce que

$$(2) \quad F_{10} = F_{20}.$$

On vérifie que dans les graphes G_1 et G_2 ils existent deux circulations standard admissibles.

Proposition 1. Réciproquement, deux circulations standards admissibles $(\bar{b}/c', F')$ dans le graphe G_1 et $(c'/\bar{b}, F'')$ dans le graphe G_2 (Fig. 3) peuvent être étendus à une circulation admissible $(\bar{b}/c'/\bar{c}'/\bar{b}, F)$ d'un graphe escalier, si $F'_{10} = F''_{20}$.

Démonstration. Nous transformons le graphe G' (Fig. 3) en le graphe G (Fig. 1) comme suit: nous détachons l'arête $P_{10}(P_{20})$ dans le sommet $y(x)$ et nous le collons dans le sommet $x(y)$.

On obtient le graphe G (Fig. 1) avec circulation admissible $F_{10} = F'_{10} = F''_{20} = F_{20}$, $F = F'$ par les arêtes de G_1 et $F = F''$ par les arêtes de G_2 .

Théorème 2. Pour toute circulation admissible de type $\bar{b}/c'/\bar{c}'/\bar{b}$ dans un graphe escalier (Fig. 1) il existe une circulation admissible de type $\bar{b}/\bar{c}'$ dans un graphe éventail (Fig. 4) [2].

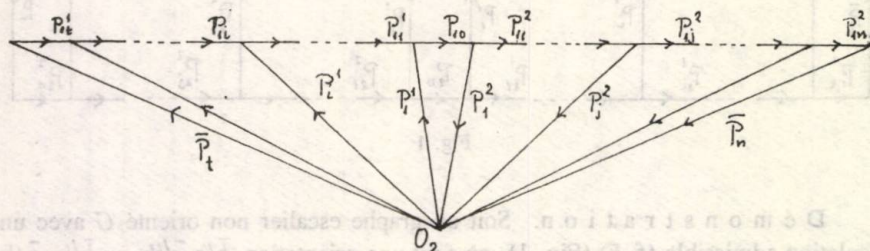


Fig. 4

Démonstration. Le graphe donné est transformée dans un graphe éventail en condensant les sommets x_{2r}^h , les arêtes P_{2r}^h avec $r = \begin{cases} i=1, t, h=1, \\ j=1, n, h=2 \end{cases}$ et P_{10}, P_{20} dans le moment O_2 , en résultant le graphe éventail $G/\bar{b}/\bar{c}'$ (Fig. 4).

Les fonctions de capacité c' et C' pour le graphe éventail sont définies ainsi:

$$(3) \quad \begin{cases} c'_{1r} = \max_k c_{kr}^k, \\ C'_{1r} = \min_k C_{kr}^k, \\ c'_r = c_r^k, \\ C'_r = C_r^k, \\ \bar{c}'_s = \bar{c}_s, \\ \bar{C}'_s = \bar{C}_s, \end{cases} \quad \begin{matrix} k=1,2, \\ r = \begin{cases} i=\overline{1,t}, \\ j=\overline{1,n}; \end{cases} \\ s=t,n; \end{matrix} \quad \begin{matrix} c'_{10} = \max(c_{10}, c_{20}), \\ C'_{10} = \min(C_{10}, C_{20}). \end{matrix}$$

On définit une circulation admissible $(\bar{b}\bar{c}', F')$ dans le graphe $G'/\bar{b}\bar{c}'$ comme la restriction de la fonction de circulation admissible F dans le graphe donné.

Pour toute arête du graphe éventail $F' = F$; donc les équations de conservation sont satisfaites.

Dans le sommet O_2 nous avons [5]

$$(4) \quad \bar{F}_t + \sum (F_i^1; 1 \leq i \leq t) = \sum (F_j^2; 1 \leq j \leq n) + \bar{F}_n.$$

Proposition 2. *Réciproquement, une circulation admissible $(\bar{b}\bar{c}', F')$ dans un graphe éventail G' (Fig. 4) peut être étendu à une circulation admissible $(\bar{b}/c', \bar{c}/b, F)$ d'un graphe escalier.*

Démonstration. Nous scindons le sommet O_2 dans les sommets

$$x_{2r}^h, \quad r = \begin{cases} i=\overline{0,t}, & h=1, \\ j=\overline{0,n}, & h=2, \end{cases} \text{ puis nous construisons les arêtes } P_{2r}^h = [x_{2r+1}^h, x_{2r}^h], \\ r = \begin{cases} i=\overline{1,t}, & h=1, \\ j=\overline{1,n}, & h=2 \end{cases} \text{ et } P_{20} = [x_{20}^1, x_{20}^2].$$

Nous définissons les fonctions de capacité sur les nouvelles arêtes:

$$(5) \quad c_{2r}^h = 0, \quad C_{2r}^h = \infty, \quad c_{20} = 0, \quad C_{20} = \infty, \quad r = \begin{cases} i=\overline{1,t}, & h=1, \\ j=\overline{1,n}, & h=2. \end{cases}$$

On peut facilement vérifier que pour une circulation admissible $(\bar{b}/c' \bar{c}'/b, F)$ dans le graphe escalier

$$(6) \quad \begin{matrix} F_{20} = F_{10}^1, \\ F_{2r}^h = F_{1r}^h = F_{1r}^h, \end{matrix} \quad r = \begin{cases} i=\overline{1,t}, & h=1, \\ j=\overline{1,n}, & h=2. \end{cases}$$

Corollaire. *Pour toute circulation admissible de type $\bar{b}/c/\bar{c}/b$ dans un graphe escalier (Fig. 1) il existe une circulation admissible de type $\bar{c}/\bar{b}$ dans un graphe éventail (Fig. 5).*

On peut démontrer le corollaire en condensant les sommets x_{1r}^h , $r = \begin{cases} i=\overline{0, t}, h=1, \\ j=\overline{0, n}, h=2, \end{cases}$ les arêtes P_{1r}^h , $r = \begin{cases} i=\overline{1, t}, h=1, \\ j=\overline{1, n}, h=2 \end{cases}$, et l'arête P_{10} dans le sommet O_1 .

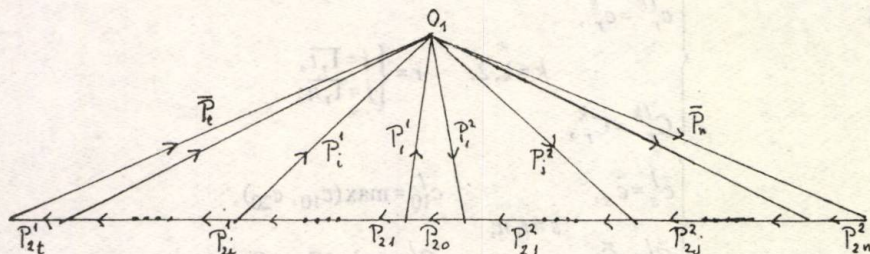


Fig. 5

Proposition 3. *Réciproquement, une circulation admissible $(\bar{c}/\bar{b}, F')$ dans le graphe éventail G (Fig. 5) peut être étendu à une circulation admissible $(\bar{b}/c/\bar{c}/b, F)$ d'un graphe escalier.*

Démonstration. Nous scindons le sommet O_1 dans les sommets x_{1r}^h , $r = \begin{cases} i=\overline{0, t}, h=1, \\ j=\overline{0, n}, h=2, \end{cases}$ puis nous construisons les arêtes $P_{1r}^h = [x_{1r+1}^h, x_{1r}^h]$, $r = \begin{cases} i=\overline{1, t}, h=1, \\ j=\overline{1, n}, h=2 \end{cases}$, et $P_{10} = [x_{10}^1, x_{10}^2]$.

Nous définissons les fonctions de capacité sur les nouvelles arêtes:

$$(7) \quad c_{10}=0, C_{10}=\infty, c_{1r}^h=0, C_{1r}^h=\infty, r = \begin{cases} i=\overline{1, t}, h=1, \\ j=\overline{1, n}, h=2. \end{cases}$$

On peut facilement vérifier qu'une circulation admissible $(\bar{b}/c/\bar{c}/b, F)$ dans le graphe escalier vérifie

$$(8) \quad F_{10}=F_{20}, \quad F_{1r}^h=F_{2r}^h=F_{2r}^h, \quad r = \begin{cases} i=\overline{1, t}, h=1, \\ j=\overline{1, n}, h=2. \end{cases}$$

Théorème 3. *Pour toute circulation admissible de type $\bar{b}/c/\bar{c}/b$ dans un graphe escalier (Fig. 1) il existe une circulation admissible dans les graphes circuits.*

Démonstration. Le graphe donné peut être écrit comme somme

directe de deux graphes escalier disjointes-sommets (Théorème 1). En vertu du Théorème 2 [4] il résulte le Théorème 3.

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Technical University "Gh. Asachi", Jassy,
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TIPURI DE CIRCULAȚII ADMISIBILE ÎNTR-UN GRAF SCARĂ NEORIENTAT (II)

(Rezumat)

Se studiază descompunerea unei circulații admisibile de tip $(\bar{b}/c' \quad c'/b, F)$ dintr-un graf scară în circulații standard admisibile din grafuri de aceeași formă și reciproc. De asemenea sus amintita circulație admisibilă este transformată în circulații admisibile de tip $\bar{b}c' \quad (c'b)$ din grafuri evantai și în circulații admisibile de tip circuit.

interne de deux graphes écartés énoncés (Théorème 1). En vertu du Théorème 2 [4] il résulte le Théorème 3.

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 Centre de Mathématiques

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TIPODI DE CIRCULATIE ADMISIBILE ÎNTR-UN GRAF SCARĂ KNOTANTĂ ÎN

(Reçu le 15 mai 1977)

Se stabilesc descompunerile unei circulații admisibile de tip (a, b) în două tipuri de circulații admisibile de tip (a, b) și (b, a). Se stabilesc descompunerile unei circulații admisibile de tip (a, b) în două tipuri de circulații admisibile de tip (a, b) și (b, a). Se stabilesc descompunerile unei circulații admisibile de tip (a, b) în două tipuri de circulații admisibile de tip (a, b) și (b, a).

ON THE BENDING OF PLATES WITH MODERATE THICKNESS MADE FROM A MATERIAL WITH VOIDS

BY

ARISTOTEL MANOLACHI

0. Introduction. In a previous paper [2] we presented a theory of thin plate with voids in which the effects of transverse shearing forces and rotatory inertia are ignored.

In what follows we propose ourselves to derive an improved plate theory involving just these effects. This theory may be available to describe the linear bending of the plates with moderate thickness made from materials with voids. In the classical elasticity such theories were already been considered under various formulations by numerous authors. For references we cite the works [1], [3], [5].

1. Fundamental Equations. The plate is referred to a cartesian system of coordinates Ox_i , $i=1, 2, 3$, with Ox_3 — axis normal to the median section. The material of the plate is assumed to be homogeneous and isotropic with small distributed voids [4]. The thickness of the plate is constant and equals $2h$.

We suppose that upon the plate faces are given conditions of the form

$$\sigma_{33}(x_1, x_2, \pm h, t) = \pm \frac{1}{2}p(x_1, x_2, t), \quad \sigma_{\gamma 3}(x_1, x_2, \pm h, t) = 0, \quad (1.1)$$

$$h_3(x_1, x_2, \pm h, t) = 0,$$

where σ_{ij} are the usual stress components and h_j — the components of the equilibrated stress. The function $p(x_1, x_2, t)$ is known.

The fundamental equations characterising a medium with voids in the presence of body loads b_δ , b_3 , l , read as follows:

i) *Equations of Motion*

$$\begin{aligned}
 (1.2) \quad & \sigma_{\gamma\delta,\gamma} + \sigma_{\delta 3,3} + \rho b_{\delta} = \rho \ddot{u}_{\delta}, \\
 & \sigma_{\gamma 3,\gamma} + \sigma_{33,3} + \rho b_3 = \rho \ddot{u}_3, \quad \gamma, \delta = 1, 2; \\
 & h_{\gamma,\gamma} + h_{3,3} + g + \rho l = \rho k_0 \ddot{\varphi},
 \end{aligned}$$

ii) Constitutive Equations

$$(1.3) \quad \sigma_{\gamma\delta} = (\lambda\theta + \beta\varphi)\delta_{\gamma\delta} + 2\mu\epsilon_{\gamma\delta}, \quad \delta_{\gamma\epsilon} = \begin{cases} 0, & \gamma \neq \delta, \\ 1, & \gamma = \delta, \end{cases}$$

$$(1.4) \quad \sigma_{\gamma 3} = 2\mu\epsilon_{\gamma 3}, \quad \theta = \theta_1 + \epsilon_{33}, \quad \theta_1 = \epsilon_{11} + \epsilon_{22},$$

$$(1.5) \quad \sigma_{33} = (\lambda\theta + \beta\varphi) + 2\mu\epsilon_{33}, \quad 2\epsilon_{ij} = u_{i,j} + u_{j,i}, \quad i, j = 1, 2, 3;$$

$$(1.6) \quad h_{\gamma} = \alpha\varphi_{,\gamma}, \quad h_3 = \alpha\varphi_{,3},$$

$$(1.7) \quad g = -\omega\dot{\varphi} - \xi\varphi - \beta\theta,$$

where u_j are the displacement components, φ — the change in volume fraction and k_0 , α , β , λ , μ , ω , ξ — the material constants.

By substituting σ_{33} from (1.5) into (1.3), we obtain

$$(1.8) \quad \sigma_{\gamma\delta} = \frac{2\mu}{\lambda + 2\mu}(\lambda\theta_1 + \beta\varphi)\delta_{\gamma\delta} + \frac{\lambda}{\lambda + 2\mu}\sigma_{33}\delta_{\gamma\delta} + 2\mu\epsilon_{\gamma\delta}.$$

In order to describe the stress state and deformation throughout the plate, we define the following measures of stresses and strains:

$$(1.9) \quad M_{\gamma\delta} = \int_{-h}^h x_3 \sigma_{\gamma\delta} dx_3, \quad Q_{\gamma} = \int_{-h}^h \sigma_{\gamma 3} dx_3, \quad H_{\gamma} = \int_{-h}^h x_3 h_{\gamma} dx_3,$$

$$E_{\gamma\delta} = \frac{3}{2h^3} \int_{-h}^h x_3 \epsilon_{\gamma\delta} dx_3, \quad E_{\gamma 3} = \frac{1}{2h} \int_{-h}^h \epsilon_{\gamma 3} dx_3,$$

$$(1.10) \quad \chi_{\gamma} = \frac{15}{8h^3} \int_{-h}^h x_3 \varphi_{,\gamma} dx_3, \quad \chi_3 = \frac{3}{4h} \int_{-h}^h \varphi_{,3} dx_3.$$

2. Basic Hypotheses and Consequences. To establish the field equations of the present theory we adopt the following hypotheses:

- there are no extensional stresses parallel to the median section;
- the integrals containing $x_3\sigma_{33}$ in the stress-strain relations (1.8) are dropped;
- the component h_3 of the equilibrated stress vector has a parabolic distribution across the plate thickness, namely

$$(2.1) \quad h_3(x_1, x_2, x_3, t) = \left[1 - \frac{x_3^2}{h^2} \right] H(x_1, x_2, t),$$

where $H(x_1, x_2, t)$ will be determined below (we note that h_3 of this form satisfies automatically condition $h_3(\pm h) = 0$ (see (1.1)));

- the displacement components and change in volume fraction are of the form

$$(2.2) \quad u_1 = x_3 \psi_1, \quad u_2 = x_3 \psi_2, \quad u_3 = W, \quad \varphi = x_3 \left[1 - \frac{x_3^2}{3h^2} \right] \phi,$$

where ψ_γ , which represent the rotations of the sections normal to Ox_γ and W, ϕ are unknown functions of variables x_1, x_2, t , which must be determined.

In virtue of the geometrical relations (1.5)₂, we have

$$(2.3) \quad 2\varepsilon_{\gamma\delta} = x_3(\psi_{\gamma,\delta} + \psi_{\delta,\gamma}), \quad 2\varepsilon_{\gamma 3} = \psi_\gamma + W_{,\gamma}.$$

In addition

$$(2.4) \quad \varphi_{,\gamma} = x_3 \left[1 - \frac{x_3^2}{3h^2} \right] \phi_{,\gamma}, \quad \varphi_{,3} = \left[1 - \frac{x_3^2}{h^2} \right] \phi.$$

By using (2.3) and (2.4), from (1.10), we obtain

$$(2.5) \quad 2E_{\gamma\delta} = \psi_{\gamma,\delta} + \psi_{\delta,\gamma}, \quad 2E_{\gamma 3} = \psi_{,\gamma} + W_{,\gamma}, \quad \chi_\gamma = \phi_{,\gamma}, \quad \chi_3 = \phi.$$

When (2.1) is substituted into (1.6)₃ with the aid of (2.4)₂, we get

$$(2.5)' \quad H(x_1, x_2, t) = \alpha \phi(x_1, x_2, t).$$

By multiplying (1.8) with x_3 , integrating the results across the plate thickness, using the hypotheses a) and b) as well as the relations

$$\frac{\lambda}{\lambda + 2\mu} = \frac{\nu}{1 - \nu}, \quad \frac{2\mu}{\lambda + 2\mu} = \frac{1 - 2\nu}{1 - \nu}, \quad \frac{2\mu}{1 - \nu} = \frac{E}{1 - \nu^2},$$

we obtain (with (1.9))

$$(2.6) \quad M_{\gamma\delta} = D[(1 - \nu)E_{\gamma\delta} + \nu E_1 \delta_{\gamma\delta}] + \beta_1 \phi \delta_{\gamma\delta}, \quad E_1 = E_{11} + E_{22},$$

where

$$D = \frac{2Eh^3}{3(1 - \nu^2)}, \quad \beta_1 = \frac{16h^3 \mu \beta}{15(\lambda + 2\mu)}$$

(E — the Young modulus, ν — the Poisson ratio).

On the basis of (1.10) and the hypothesis c), from (1.6) we find

$$(2.7) \quad H_\gamma = \alpha \chi_\gamma, \quad H_3 = \alpha \chi_3.$$

Also, from (1.4), we get

$$(2.8) \quad Q_\gamma = 2\mu h E_{\gamma 3}.$$

Finally, when (2.5) are substituted into (2.6) — (2.8), we obtain

$$(2.9) \quad \begin{aligned} M_{\gamma\delta} &= D[(1 - \nu)\psi_{\gamma,\delta} + \nu\psi_{\rho,\rho}\delta_{\gamma\delta}] + \beta_1 \phi \delta_{\gamma\delta}, \\ Q_\gamma &= 2\mu h(\psi_\gamma + W_{,\gamma}), \end{aligned}$$

$$H_\gamma = \alpha \phi_{,\gamma}, \quad H_3 = \alpha \phi.$$

3. The Equations of Motion. Following the classical procedure and (1.1), from (1.2) we obtain

$$(3.1) \quad M_{\gamma\delta,\delta} - Q_{\gamma} + \rho B_{\gamma} = \frac{2\rho h^3}{3} \ddot{\psi}_{\gamma},$$

$$Q_{\gamma,\gamma} + p + \rho B_3 = 2\rho h \ddot{W},$$

where

$$B_{\gamma} = \int_{-h}^h x_3 b_{\gamma} dx_3, \quad B_3 = \int_{-h}^h b_3 dx_3.$$

Taking into account $h_3(\pm h) = 0$, from (1.2)₄ we get

$$(3.2) \quad H_{\gamma,\gamma} - \int_{-h}^h h_3 dx_3 + \int_{-h}^h x_3 g dx_3 + \rho L = \frac{8\rho k_0 h^3}{15} \ddot{\phi},$$

with

$$L = \int_{-h}^h x_3 l dx_3.$$

In view of (1.7), (2.1) and (2.5)', we have

$$(3.3) \quad \int_{-h}^h h_3 dx_3 = \frac{4\alpha h}{3} \phi(x_1, x_2, t),$$

$$(3.4) \quad \int_{-h}^h x_3 g dx_3 = -\frac{8h^3}{15} (\omega \dot{\phi} + \xi_1 \phi) - \frac{2h^3}{3} \beta_1^* \psi,$$

with

$$\xi_1 = \xi - \frac{\beta^2}{\lambda + 2\mu}, \quad \beta_1^* = \frac{2\mu\beta}{\lambda + 2\mu}, \quad \psi = \psi_{1,1} + \psi_{2,2}.$$

Finally by inserting (3.3) and (3.4) into (3.2) with the aid of (2.9)_{1,2}, we arrive to the differential equation

$$(3.5) \quad \alpha \Delta \phi - \eta \phi - \beta_2 \psi - \omega \dot{\phi} + \rho L = \rho k_0 \ddot{\phi},$$

where the constants η and β_2 are given by

$$\eta = \xi_1 + \frac{5\alpha}{2h^2}, \quad \beta_2 = \frac{5}{4}\beta_1^* = \frac{5\mu\beta}{2(\lambda + 2\mu)}.$$

On substituting (2.6) and (2.8) into (3.1) we obtain the equation of motion expressed in terms of ψ_{γ} , ϕ and W , as follows

$$(3.6) \quad \frac{1}{2} D[(1-\nu) \Delta \psi_{\gamma} + (1+\nu) \psi_{\gamma,\gamma}] - \mu h (\psi_{\gamma} + W_{,\gamma}) + \\ + \beta_1 \phi_{,\gamma} + \rho B_{\gamma} = \frac{2}{3} \rho h^3 \ddot{\psi}_{\gamma},$$

$$\mu h(\Delta W + \psi) + p + \rho B_3 = 2h\rho \ddot{W}.$$

The system of equations (3.6) and (3.5) involves both the effects of transverse shear deformations and rotatory inertia. The classical counterpart of these equations are that derived by Mindlin [1]. Of course, the equation (3.5) does not appear in the classical formulation of the plate theory.

4. Initial and Boundary Conditions. To the system of equations (3.5) and (3.6) we must associate some initial and boundary conditions which assign the unicity of the solution for a given initial and boundary data. Let Δ be the domain of the median section of the plate and C the frontiere of Δ (the edge curve). We assume that C is regular of class C^1 and denote by n_γ the components of the outward normal at C .

The initial and boundary conditions are of the form:

a) *initial conditions* (at $t=0$)

$$(4.1) \quad \begin{aligned} \psi_\gamma(P, 0) &= \dot{\psi}_\gamma(P), \quad W(P, 0) = \dot{W}(P), \quad \phi(P, 0) = \dot{\phi}(P); \\ \dot{\psi}_\gamma(P, 0) &= \hat{\psi}_\gamma(P), \quad \dot{W}(P, 0) = \hat{W}(P), \quad \dot{\phi}(P, 0) = \hat{\phi}(P), \quad P \in \Delta; \end{aligned}$$

b) *boundary conditions* (at C)

$$(4.2) \quad \begin{aligned} \psi_n &= \bar{\psi}_n(s), \quad \psi_t = \bar{\psi}_t(s), \quad W = \bar{W}(s), \quad \phi = \bar{\phi}(s), \quad \text{at } \bar{C}_1; \\ M_n &= \bar{M}_n(s), \quad M_{nt} = \bar{M}_{nt}(s), \\ Q_n &= \bar{Q}_n(s), \quad H_n = \bar{H}_n(s), \quad \text{at } \bar{C}_2, \end{aligned}$$

where $\bar{C}_1 \cup \bar{C}_2 = C$, $\bar{C}_1 \cap \bar{C}_2 = \emptyset$ and s is the length of arc along C . Other notations appearing in (4.2) are

$$(4.3) \quad \begin{aligned} \psi_n &= \psi_1 n_1 + \psi_2 n_2, \quad \psi_t = \psi_2 n_1 - \psi_1 n_2, \\ M_n &= M_{11} n_1^2 + M_{22} n_2^2 + 2M_{12} n_1 n_2, \\ M_{nt} &= (M_{22} - M_{11}) n_1 n_2 + M_{12} (n_1^2 - n_2^2), \\ Q_n &= Q_1 n_1 + Q_2 n_2, \quad H_n = H_1 n_1 + H_2 n_2. \end{aligned}$$

In the next we note some special cases which are encountered in practice:

i) when $C_2 = \emptyset$, $C_1 = C$ and

$$\psi_n = 0, \quad \psi_t = 0, \quad W = 0, \quad \text{at } C,$$

we obtain the case of a plate with clamped edge;

ii) when $C_1 = \emptyset$, $C_2 = C$ and

$$M_n = 0, \quad M_{nt} = 0, \quad Q_n = 0, \quad H_n = 0 \quad \text{at } C,$$

we are in the situation of a plate with free edge;

iii) mixed boundary conditions requiring

$$W=0, \quad M_n=0, \quad M_{nt}=0, \quad H_n=0 \quad \text{at } C,$$

can be associated to the case corresponding to a plate with simply supported edge.

Remark. Under some positiveness conditions involving the material constants $\alpha, \beta, \lambda, \mu, \xi$ (see [4]), we can prove that the solutions of the problems (3.5), (3.6), (4.1) and (4.2) are unique.

5. Some Special Cases. In what follows we neglect the body forces. By differentiating the first two equations among (3.6) with respect to x_1 and x_2 respectively, summing up the results and using (3.6)₃, we obtain

$$(5.1) \quad \left[D\Delta - I \frac{\partial^2}{\partial t^2} \right] \psi + p - J \frac{\partial^2 W}{\partial t^2} + \beta_1 \Delta \phi = 0, \quad I = \frac{2\rho h^3}{3}, \quad J = 2\rho h.$$

On the other hand, according to (3.6)₃, we have

$$(5.1)' \quad \psi = \left[-\Delta + \frac{J}{\mu h} \frac{\partial^2}{\partial t^2} \right] W - \frac{p}{\mu h}.$$

By substituting (5.1)' into (5.1), one gets

$$(5.2) \quad \left[D\Delta - I \frac{\partial^2}{\partial t^2} \right] \left[-\Delta + \frac{J}{\mu h} \frac{\partial^2}{\partial t^2} \right] W - J \frac{\partial^2 W}{\partial t^2} + \beta_1 \Delta \phi =$$

$$= -p + \left[D\Delta - I \frac{\partial^2}{\partial t^2} \right] \frac{p}{\mu h}.$$

Also, by inserting the function ψ given by (5.1)' into (3.5) we obtain

$$(5.3) \quad \left[\alpha\Delta - \eta - \omega \frac{\partial}{\partial t} - K \frac{\partial^2}{\partial t^2} \right] \phi + \beta_2 \left[\Delta - \frac{J}{\mu h} \right] W = -\frac{\beta_2 p}{\mu h}, \quad K = \rho k_0.$$

1. *The terms containing the rotatory inertia are omitted.* In this case the equation (5.2) becomes

$$D\Delta \left[-\Delta + \frac{J}{\mu h} \frac{\partial^2}{\partial t^2} \right] W - J \frac{\partial^2 W}{\partial t^2} + \beta_1 \Delta \phi = -p + \frac{D}{\mu h} \Delta p$$

together with (5.3) which is not affected.

2. *The terms involving the shear effects are neglected.* Then we obtain

$$D\Delta\Delta W + J \frac{\partial^2 W}{\partial t^2} - \beta_1 \Delta \phi = p$$

and respectively

$$\left[\alpha\Delta - \eta - \omega \frac{\partial}{\partial t} - K \frac{\partial^2}{\partial t^2} \right] \phi + \beta_2 \Delta W = 0.$$

3. Finally, by disregarding the effect of voids we arrive to the classical plate

equation

$$D \Delta \Delta W + J \frac{\partial^2 W}{\partial t^2} = p(x_1, x_2, t).$$

We note that the appropriate boundary conditions in the Cases 1...3 are obtained by a common procedure from (4.1) and (4.2).

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ASUPRA ÎNCOVOIERII PLĂCILOR CU GROSIME MODERATĂ ALCĂTUITE DIN MATERIAL CU GĂURI

(Rezumat)

Se prezintă o teorie a plăcilor subțiri cu goluri în care se iau în considerație atât efectele forțelor tăietoare transversale cât și ale inerției de rotație.

ON FINITE DEFORMATION OF ELASTIC
DIPOLAR CONTINUUM

BY

GABRIELA-SILVIA CIUMAȘU

In [2] are established the basic equations for dipolar elastic solid.

Using *Stoppelli's* method we prove here a theorem of existence and uniqueness for the finite deformation of dipolar elastic solid, when the magnitudes of the external loads are proportional to the real parameter ε .

We denote by D a homogeneous reference configuration of an elastic body, and by Σ the boundary of D . We refer the motion of continuum medium to a fixed rectangular cartesian coordinate system, with the origin O belonging to D , and denote by X_K, x_i the material and spatial coordinates respectively.

The motion of a continuum is represented [2] by

$$(1) \quad x_i^* = x_i(X_K, t), \quad x_{iK}^* = x_{iK}(X_K, t),$$

where x_{iK}^* are the components of a tensor, which is called a dipolar displacement tensor.

We consider a family of problems depending upon a parameter ε ; the loads are assumed to be of the form

$$\rho_0 \varepsilon F_i^{(1)}, \quad \rho_0 \varepsilon F_{iK}^{(1)}, \quad \varepsilon T_i^{(1)}, \quad \varepsilon S_{iK}^{(1)},$$

where $F_i^{(1)}, F_{iK}^{(1)}, T_i^{(1)}, S_{iK}^{(1)}$ are prescribed functions.

In this situation we have the basic equations of equilibrium

$$T_{Ki,K}^* + \rho_0 \varepsilon F_i^{(1)} = 0, \quad S_{LiK,L}^* + \rho_0 \varepsilon F_{iK}^{(1)} - x_{i,A}^* \bar{S}_{AK}^* = 0,$$

$$T_{Ki}^* N_K = \varepsilon T_i^{(1)}, \quad S_{LiK}^* N_L = \varepsilon S_{iK}^{(1)},$$

(2)

$$\sigma = \sigma(E_{AB}^*, E_{B:A}^*, E_{B:AK}^*),$$

$$S_{AB}^{*/} = T_{AB}^{*} - X_{B,i} (\bar{S}_{AC}^{*} x_{iC}^{*} + S_{KAC}^{*} x_{iC}^{*}, K) = \frac{\partial \sigma}{\partial E_{AB}^{*}},$$

$$\bar{S}_{AB}^{*} = \frac{\partial \sigma}{\partial E_{B:A}^{*}}, \quad S_{KAB}^{*} = \frac{\partial \sigma}{\partial E_{B:AK}^{*}}.$$

We have used the notations from [2].

We shall prove that the system (2) has a unique solution $z^{*}(x_i^{*}, x_{iA}^{*})$ which satisfies the condition

$$(3) \quad x_i^{*}(0) = 0, \quad x_{iA}^{*}(0) = X_{iA}^{*}(0).$$

If we superimpose a rigid rotation Q about O upon a deformation $z^{*}(x_i^{*}, x_{iA}^{*})$ we obtain a new deformation z defined by

$$(4) \quad x_i = Q_{ij} x_j^{*}, \quad x_{iA} = Q_{ij} x_{jA}^{*}.$$

We can prove

Theorem 1. *The functions $z^{*}(x_i^{*}, x_{iA}^{*})$ are a solution of problem (2) if and only if $z(x_i, x_{iA})$ defined by (4) are a solution of the system*

$$(5) \quad \begin{aligned} T_{Ki,K} + \rho_0 \varepsilon Q_{iS} F_S^{(1)} &= 0, \\ S_{LiK,L} + \rho_0 \varepsilon Q_{iR} F_{rK}^{(1)} - x_{i,A} \bar{S}_{AK}^{(1)} &= 0, \\ T_{Ki} N_K &= \varepsilon Q_{iS} T_S^{(1)}, \quad S_{LiK} N_L = \varepsilon Q_{iR} S_{rK}^{(1)}. \end{aligned}$$

We prefer to consider the system (5) and to determine the rigid rotation Q so as to make (5) have a unique solution z such that (3) holds. Then (2) has a unique solution $z^{*} = Q^T z$ which also satisfies (3).

We denote by Z the set of elements

$$z = \{z_M\}_{M \in (1, \dots, 12)} = (u_K, u_{eA})_{K, e, A \in (1, 2, 3)},$$

$$z \in C^{2,\lambda}(\bar{D}), \quad z(0) = (0, \dots, 0).$$

For $z \in Z$ we introduce the norm

$$(6) \quad \|z\| = \sum_{M=1}^{12} \left\{ \max |z_M| + \sum_{j=1}^3 \max |z_{M,j}| + \sum_{j,r=1}^3 \max |z_{M,rj}| + \beta_{Mjr} \right\},$$

where

$$\{\beta_{Mjr}\}_{\substack{M \in (1, \dots, 12) \\ \delta, r \in (1, 2, 3)}} = \{\beta_{kjr}, \beta_{LAjr}\}_{k,j,r,L,A \in (1, 2, 3)}$$

are the Hölder-coefficients of z_{kjr} . With (6) as norm Z becomes a Banach space.

We denote by Z' the class of elements $z' = \{\alpha_i, \eta_{iK}, \beta_i, \theta_{iK}\}$ with the following properties:

$$a_1) \alpha_i, \eta_{iK} \in C^{0,\lambda}(D), \quad \beta_i, \theta_{iK} \in C^{1,\lambda}(\Sigma);$$

$$\int_D \alpha_i dv + \int_\Sigma \beta_i dA = 0,$$

b₁)

$$\int_D \varepsilon_{ijr}(X_j \alpha_r + X_{jC} \eta_{rC}) dv + \int_\Sigma \varepsilon_{ijr}(X_j \beta_r + X_{jC} \theta_{rC}) dA = 0.$$

For $z' \in Z'$ we define the norm

$$(7) \quad \|z'\| = \sum_{i=1}^3 \left(\max_{X \in D} |\alpha_i| + \max_{X \in \Sigma} |\beta_i| + b_i + C_i \right) + \sum_{i,j=1}^3 \left(\max_{X \in D} |\eta_{ij}| + \max_{X \in \Sigma} |\theta_{ij}| + A_{ij} + B_{ij} \right),$$

where b_i, A_{ij} are the Hölder-coefficients of α_i, η_{ij} and C_i, B_{ij} are the greatest of the Hölder-coefficients of the derivatives β_i, θ_{ij} with respect to the parameters of some parametric representation of Σ .

It can be verified that Z' with (7) as norm is a Banach space.

We denote by Z'' the subspace of Z obtained by adding the requirements that

$$u_{k,r}(0) - u_{r,k}(0) = 0, \quad z(0) = (0, 0, \dots, 0).$$

It can be verified that Z'' is also a Banach space. We assume that:

a₂) D is a configuration of natural equilibrium and the region occupied by the body is compact.

b₂) The boundary Σ has Hölder continuous principal curvatures with exponent λ and it is possible to cover Σ by a finite number of regions Σ^p , each of which can be represented parametrically on a disc C^p so as every point of Σ be interior to at least one Σ^p .

c₂) The strain energy function together with its derivatives up to the fourth order are continuous with respect to $u_{i,L}, u_{iA}, u_{iA,K}$ when they vary within the sphere $\|z\| < R$ of the space Z .

d₂) The function $\sigma = \sigma(\varepsilon_{ij}, \nu_{ij}, \nu_{kij})$ satisfies the condition

$$\frac{\partial^2 \sigma}{\partial A_{ML}^\alpha \partial A_{NP}^\beta} \mu_L \mu_\beta B_M^\alpha B_N^\beta > 0 \quad (\alpha, \beta = 1, 2),$$

for all non-vanishing vectors $\mu = \{\mu_L\}_{L \in (1, 2, 3)}$ and

$$B = \{B_M\}_{M \in (1, \dots, 12)};$$

$$A_{ML}^{(1)} = \varepsilon_{iL}, \quad A_{ML}^{(2)} = \nu_{iL} \quad \text{for } M, i, L \in (1, 2, 3),$$

$$A_{ML}^{(1)} = A_{ML}^{(2)} = \nu_{KiL} \quad \text{for } M \leftrightarrow (K, i), \quad M \in (4, \dots, 12), \quad K, i, L \in (1, 2, 3).$$

$$e_2) \quad z'_0 = \{\rho_0 F_i^{(1)}, \rho_0 F_{iK}^{(1)}, T_i^{(1)}, S_{iK}^{(1)}\} \in Z'.$$

We prove that the system (5) has a unique solution $z \in C^{2, \lambda}(\bar{D})$ which satisfies the condition $z(0) = (0, 0, \dots, 0)$.

Let us consider $z \in Z$ a given deformation and introduce the additional load $(\alpha_i, \eta_{iK}, \beta_i, \theta_{iK})$,

$$(8) \quad \begin{aligned} T_{Ki, K} + \rho_0 \varepsilon Q_{iS} F_S^{(1)} &= \alpha_i, \\ S_{LiK, L} + \rho_0 \varepsilon Q_{iS} F_{rK}^{(1)} - x_{i, A} \bar{S}_{AK} &= \eta_{iK}, \\ -T_{Ki} N_K + \varepsilon Q_{iS} T_S^{(1)} &= \beta_i, \quad -S_{LiK} N_L + Q_{iS} S_{rK}^{(1)} = \theta_{iK}. \end{aligned}$$

First we shall determine the rigid rotation Q in such a way that the loads $z' \in Z'$, for given z .

The hypotheses (a_1) are obvious. It is easy to show that the relation $(b_1)_1$ is satisfied. The condition $(b_1)_2$ may be written as

$$(9) \quad \begin{aligned} \varepsilon \varepsilon_{ijr} Q_{rS} A_{jS}^{(R)/} &= M_i, \\ \sum_{j=1}^3 Q_{ji} Q_{Kj} &= \delta_{jK}, \quad \det(Q_{Kj}) = 1, \end{aligned}$$

where $A_{jS}^{(1)}$ is the astatic load of the forces defined in [1]

$$A_{jS}^{(1)} = \int_D (\rho_0 X_j F_r^{(1)} + \rho_0 X_j C F_{rC}^{(1)}) dv + \int_\Sigma (X_j T_r^{(1)} + X_j C S_{rC}^{(1)}) dA$$

and

$$(10) \quad \begin{aligned} M_i &= \varepsilon_{ijr} \int_D \left\{ u_{r, A} (\delta_{jB} S'_{BA} X_{jC, L} S_{LAC} + X_{jC} \bar{S}_{AC}) + \right. \\ &\quad \left. + \delta_{jA} \bar{S}_{AC} u_{rC} + \delta_{jA} S_{KAC} u_{rC, K} \right\} dv. \end{aligned}$$

In the same manner as in [4] one proves the following

Theorem. If the loads $\{\rho_0 F_i^{(1)}, \rho_0 F_{iK}^{(1)}, T_i^{(1)}, S_{iK}^{(1)}\}$ do not possess an axis of equilibrium, then it is possible to find positive numbers β_1, α_1 such that for $\beta < \beta_1$ to find that system (9) admits a unique solution satisfying the relation

$$|Q_{ij} - \delta_{ij}| < \beta,$$

when $|\vec{M}| < \alpha |\varepsilon|$ and $Q_{ij} = \delta_{ij}$ for $M_i = 0$.

Proof. It is easily shown that

$$(11) \quad \begin{aligned} |S'_{AB}(z)| &\leq K_1 \|z\|, & |S_{LPK}(z)| &\leq K_2 \|z\|, \\ |\bar{S}_{PB}(z)| &\leq K_3 \|z\|, \end{aligned}$$

where

$$\begin{aligned} K_1 &= \max \left\{ \left| \frac{\partial S'_{AB}(z)}{\partial u_{i,K}} \right|, \left| \frac{\partial S'_{AB}(z)}{\partial u_{iK}} \right|, \left| \frac{\partial S'_{AB}(z)}{\partial u_{iA,K}} \right| \right\}, \\ K_2 &= \max \left\{ \left| \frac{\partial S_{LPK}(z)}{\partial u_{i,K}} \right|, \left| \frac{\partial S_{LPK}(z)}{\partial u_{iK}} \right|, \left| \frac{\partial S_{LPK}(z)}{\partial u_{ik,L}} \right| \right\}, \\ K_3 &= \max \left\{ \left| \frac{\partial \bar{S}_{PC}(z)}{\partial u_{i,A}} \right|, \left| \frac{\partial \bar{S}_{PC}(z)}{\partial u_{iA}} \right|, \left| \frac{\partial \bar{S}_{PC}(z)}{\partial u_{iA,B}} \right| \right\}. \end{aligned}$$

From (10) and (11) we obtain

$$(12) \quad \|\vec{M}\| \leq K \|z\|^2,$$

where

$$K = L \text{vol} D, \quad L = \max(K_1, K_2, K_3, |X_{jC}|, |X_{jC,L}|).$$

We take

$$(13) \quad \|z\| < \alpha_1 \sqrt{|\varepsilon|}, \quad \alpha_1 = \frac{\sqrt{\alpha}}{K},$$

$$(14) \quad |\varepsilon| < \frac{R^2}{\alpha_1^2}.$$

If (14) holds, then there is a unique orthogonal tensor Q such that (9) are satisfied. With this choice of Q , the load z determined by (8) is equilibrated. In view of the hypotheses, it defines a point in the Banach space Z and then (8) can be seen to define a mapping of a neighbourhood Λ of $O \in \Sigma$ into the space Σ' .

$$(15) \quad G(z) = Z'.$$

According to a known inverse mapping theorem of functional analysis, the mapping G is locally invertible in a neighbourhood of $O \in Z$ if the Fréchet differential $\delta G(O)$ is a invertible linear transformation of Z into Z' .

The mapping G is Gateaux differentialble and we can prove that

$$(16) \quad \|\delta G(z_0, z)\| \leq M_1 \|z\|,$$

$$\|\delta G(z_1, z) - \delta G(z_2, z)\| \leq M_2 \|z_1 - z_2\| \|z\| |\varepsilon|^{-1/2}.$$

Thus, the mapping G is Fréchet differentiable.

We can show that $\delta Z' = D(z_0, \delta z)$ coincides with the fundamental linear system of dipolar medium

$$\begin{aligned}\delta \alpha_A(0, z) &= [\delta T_{BA}(0, z)]_B, \quad \delta \eta_{AB} = [\delta S_{KAB}(0, z)]_K - \delta \bar{S}_{AB}(0, z), \\ \delta \beta_A(0, z) &= -\delta T_{BA}(0, z) N_B, \quad \delta \theta_{AB}(0, z) = -\delta S_{KAB}(0, z) N_B.\end{aligned}$$

One proves that the solution has Hölder-continuous second derivatives with exponent λ and the components of the displacement satisfy (6).

Theorem. *If the hypotheses are satisfied and the loads $\{\rho_0 F_i^{(1)}, \rho F_{iK}^{(1)}, T_i^{(1)}, S_{iK}^{(1)}\}$ do not possess an axis of equilibrium, then we can determine the positive numbers η, r', ε' in such a way that if $\eta < \eta'$ one can find other two numbers $r < r', \varepsilon'_0 < \varepsilon'$ so that for every $|\varepsilon| < \varepsilon'_0$ the system (5) has a unique solution $Q = (Q_{ij}), z = (u_i, u_{iA})$ satisfying the conditions*

$$|Q_{ij} - \delta_{ij}| < \eta, \quad \|z\| < r\sqrt{|\varepsilon|}.$$

Therefore the deformation z satisfies all requirements for the solution of (5), and $z^*(x_i^*, x_{iA}^*)$ with $x_i^* = Q_{ji}x_j, x_{iA}^* = Q_{ji}x_{jA}$ is the solution of the original problem defined by (2).

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ASUPRA DEFORMĂȚILOR FINITE ALE MEDIILOR CONTINUE ELASTICE DIPOLARE

(Rezumat)

Utilizând metoda dată de Stoppelli se demonstrează o teoremă de existență și unicitate pentru deformările finite ale unui solid elastic dipolar când mărimea încărcărilor finite exterioare este proporțională cu un parametru real.

MINIMUM PRINCIPLES IN A GRADE CONSISTENT MICROPOLAR THEORY OF PIEZOELECTRICITY WITH POLARIZATION GRADIENT

BY

D. VIERU

1. Introduction. In [7] we derived a generalized linear theory for micropolar nonsimple piezoelectric material with polarization gradient into a quasistatic electric field. In [2] with the method suggested by [4], we establish a reciprocal relation and new uniqueness results.

The purpose of this paper is to prove minimum principles in the theory of piezoelectricity.

2. Quasistatic Electric Fields. We assume that the body occupies a bounded region B with Lipschitz boundary S which consists of a finite numbers of smooth surfaces. Let γ_α be intersections of two adjoined smooth surfaces and $\Gamma = \bigcup_\alpha \gamma_\alpha$. We refer the motion of the continuum to a fixed system rectangular Cartesian axes Ox_i ($i=1, 2, 3$). Let the outward unit normal at S be n_i , referred to our fixed orthogonal frame of reference. The fundamental system of field equations consists of the equations of motion [2], [7]

$$(2.1) \quad \begin{aligned} t_{ji,j} + F_i &= \rho \ddot{u}_i, \quad t_{ji} = \tau_{ji} - \mu_{kji,k}, \\ \mu_{ji,j} + \varepsilon_{ijk} \tau_{jk} + G_i &= I_{ij} \ddot{\varphi}_j, \quad \text{in } B; \end{aligned}$$

and the equations of the quasistatic electric field

$$(2.2) \quad \begin{aligned} \pi_{ji,j} + \pi_i - \varphi_{,i} + E_i^0 &= 0, \\ P_{i,i} - \varepsilon_0 \varphi_{,ii} &= f, \\ E_i &= -\varphi_{,i}, \quad \text{in } B. \end{aligned}$$

In this equations we have used the following notations: u_i — components of the displacement vector; φ_i — components of micro-rotation vector; P_i — polarization; φ — electric potential; F_i — components of the body force vector; G_i — components of the body-couple vector; E_i^0 — electric field; f — volume density of free charge;

τ_{ij} — components of the stress tensor; μ_{ij} — components of the couple-stress tensor; μ_{ijk} — components of the hyperstress tensor; π_i — effective local electric force; π_{ij} — components of the dipolar electric tensor; ρ — mass density; I_{ij} — coefficients of microinertia; ε_{ijk} — the alternating tensor.

We consider the following boundary conditions:

$$\begin{aligned}
 (2.3) \quad & u_i = \tilde{u}_i \text{ on } S_1 \times I, \quad A_i = \tilde{A}_i \text{ on } S_2 \times I, \\
 & \varphi_i = \tilde{\varphi}_i \text{ on } S_3 \times I, \quad m_i = \mu_{ji} n_j = \tilde{m}_i \text{ on } S_4 \times I, \\
 & Du_i = u_{i,r} n_r = \tilde{g}_i \text{ on } S_5 \times I, \quad B_i = \tilde{B}_i \text{ on } S_6 \times I, \\
 & \varphi = \tilde{\varphi} \text{ on } S_7 \times I, \quad (\varepsilon_0 < \varphi_{,i} > -P_i) n_i = f_1 \text{ on } S_8 \times I, \\
 & p_i = \pi_{ji} n_j = 0 \text{ on } S \times I, \quad P_i = \tilde{P}_i \text{ on } S \times I, \\
 & C_i = \tilde{C}_i \text{ on } \Gamma \times I,
 \end{aligned}$$

where $S_r (r=1, 2, \dots, 8)$ denote subsets of S so that:

$S_1 \cup S_2 \cup \Gamma = S_3 \cup S_4 \cup \Gamma = S_5 \cup S_6 \cup \Gamma = S_7 \cup S_8 \cup \Gamma = S$, $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = \emptyset$, $I = [0, t_1]$, t_1 is an instant that may be infinite; $\langle f \rangle$ denotes the difference of limits from both sides of S , and $\tilde{u}_i, \tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{\varphi}_i, \tilde{m}_i, \tilde{g}_i, \tilde{\varphi}, f_1, \tilde{P}_i$ are prescribed functions. In (2.3) we have used the following notations:

$$\begin{aligned}
 (2.4) \quad & A_i = [\tau_{ji} - \mu_{kji, k} + (\mu_{pji} n_p n_r - \mu_{pri} n_p n_j)_{,r}] n_j, \\
 & B_i = \mu_{sji} n_s n_j, \quad C_i = \langle \varepsilon_{jrs} \mu_{pji} n_p n_s s_r \rangle,
 \end{aligned}$$

where $\langle h \rangle$ denotes the difference of limits from both sides of γ_α and s_i represents the unit tangent vector to γ_α .

We consider the following initial conditions:

$$\begin{aligned}
 (2.5) \quad & u_i(x, 0) = \tilde{a}_i(x), \quad \dot{u}_i(x, 0) = \tilde{b}_i(x), \\
 & \varphi_i(x, 0) = \tilde{c}_i(x), \quad \dot{\varphi}_i(x, 0) = \tilde{d}_i(x), \quad x \in \bar{B},
 \end{aligned}$$

where $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i, \tilde{d}_i$ are prescribed functions.

In [2] we proved the following reciprocal relations for piezoelectric micropolar nonsimple material:

$$\begin{aligned}
 (2.6) \quad & \int_B \left[f_i^{(1)} * u_i^{(2)} + g_i^{(1)} * \varphi_i^{(2)} + \gamma * (E_i^{0(1)} * P_i^{(2)} + f^{(1)} * \varphi^{(2)}) \right] dv + \\
 & + \int_S \left[\gamma * [A_i^{(1)} * u_i^{(2)} + B_i^{(1)} * Du_i^{(2)} + m_i^{(1)} * \varphi_i^{(2)} + f_1^{(1)} * \varphi^{(2)} - \right. \\
 & \left. - C_{ij} \mu * h_{ji}^{(2)}] da + \int_\Gamma \gamma * C_i^{(1)} * u_i^{(2)} dl =
 \end{aligned}$$

$$\begin{aligned}
&= \int_B \left[f_i^{(2)} * u_i^{(1)} + g_i^{(2)} * \varphi_i^{(1)} + \gamma * \left(E_i^{0(2)} * P_i^{(1)} + f^{(2)} * \varphi^{(1)} \right) \right] dv + \\
&+ \int_S \left[\gamma * \left[A_i^{(2)} * u_i^{(1)} + B_i^{(2)} * Du_i^{(1)} + m_i^{(2)} * \varphi_i^{(1)} + f_1^{(2)} * \varphi^{(1)} - \right. \right. \\
&\quad \left. \left. - C_{ij} \mu * h_{ji}^{(1)} \right] da + \int_\Gamma \gamma * C_i^{(2)} * u_i^{(1)} dl, \right. \\
&\int_B \left(F_i * u_i + G_i * \varphi_i + E_i^0 * P_i + f * \varphi \right) dv - \\
&- \int_B \left(\rho \ddot{u}_i * u_i + I_{ij} \ddot{\varphi}_i * \varphi_j \right) dv = \\
(2.7) \quad &= \int_B \left[\tau_{ij} * e_{ij} + \mu_{ij} * \kappa_{ij} + \mu_{ijk} * \kappa_{ijk} - \pi_i * P_i \right] dv + \\
&+ \int_B \left[\varepsilon_0 E_i * E_i + (\pi_{ij} - C_{ij}) * P_{j,i} \right] dv,
\end{aligned}$$

where

$$\begin{aligned}
f_i^{(\alpha)} &= \gamma * F_i^{(\alpha)} + \rho (t \tilde{b}_i^{(\alpha)} + \tilde{a}_i^{(\alpha)}), \\
g_i^{(\alpha)} &= \gamma * G_i^{(\alpha)} + I_{ij} (t \tilde{d}_i^{(\alpha)} + \tilde{c}_i^{(\alpha)}), \quad \alpha = 1, 2; \\
(2.8) \quad h_{ij} &= P_i n_j, \quad \gamma(t) = t, \quad \mu(t) = 1, \quad t \in I, \\
e_{ij} &= u_{j,i} + \varepsilon_{jik} \varphi_k, \quad \kappa_{ij} = \varphi_{j,i}, \quad \kappa_{ijk} = u_{k,ij}.
\end{aligned}$$

In (2.6)–(2.8) $u * v$ denotes the convolution of u and v , while C_{ij} are constitutive coefficients.

3. Minimum Principles. In [4] I e ș a n have established minimum principles for the linear theory of piezoelectricity. In this section we extend these results to the quasistatic theory for micropolar non-simple piezoelectric material with polarization gradient.

We say that ψ is bounded at infinity if $\lim_{t \rightarrow \infty} \psi(x, t)$ exists for each x in the domain of definition of ψ . Let X be the set of all functions ψ on $\bar{B} \times I$, $I = [0, \infty)$, with the following properties:

- (i) ψ is of class C^4 on $B \times (0, \infty)$ and of class $C^{3,0}$ on $\bar{B} \times I$;
- (ii) $\psi, \dot{\psi}, \psi_{,i}, \psi_{,ij}, \psi_{,ijk}$ are bounded at infinity.

We introduce the set M of admissible weight functions. We say that $g \in M$ if

g is a function on I with the following properties:

$$(\alpha) \quad \int_0^\infty \int_0^\infty g^{[k]}(t+s) dt ds \quad \text{exists for } k \geq 0,$$

$$(\beta) \quad g(t) = \int_0^\infty G(p) e^{-pt} dp, \quad t \in I,$$

where $g^{[k]}(t) = \partial^k g / \partial t^k$, and G is continuous and positive on I and has a finite limit at infinity. Let Y be the set of all continuous functions on $\bar{B} \times I$ that are bounded at infinity. Let $u, v \in Y$. We introduce the notation

$$(3.1) \quad [u, v]_g = \int_0^\infty \int_0^\infty \int_B g(t+s) u(x, t) v(x, s) dt ds dv,$$

where g is any fixed function of M . We note that

$$(3.2) \quad [u, v]_g = [v, u]_g.$$

Let A be a linear operator from X into Y . We denote by D_A the domain of definition of the operator A . We say that the operator A is g -symmetric if

$$(3.3) \quad [Au, v]_g = [u, Av]_g,$$

for any $u, v \in D_A$. A g -symmetric operator A is called positive if

$$(3.4) \quad [Au, u]_g \geq 0,$$

for any $u \in D_A$. We consider the equation

$$(3.5) \quad Au = F,$$

where $F \in Y$. Let Λ_g be the functional on D_A defined by

$$(3.6) \quad \Lambda_g\{u\} = [Au, u]_g - 2[u, F]_g,$$

for every $u \in D_A$. Let $u, u^0 \in D_A$ and define $w = u^0 - u$. Clearly, if A is g -symmetric, then

$$\Lambda_g\{u^0\} = \Lambda_g\{u\} + 2[Au - F, w]_g + [Aw, w]_g.$$

An immediate consequence of this relation is the following

Lemma 3.1. *Let A be a g -positive operator and let Λ_g be the functional on D_A defined by (3.6). Further, let u be a solution of the equation (3.5). Then*

$$\Lambda_g\{u^0\} \geq \Lambda_g\{u\},$$

for every $u^0 \in D_A$.

In what follows we denote by ψ^* the Laplace transform with respect to time of the function ψ . By (β) and (3.1)

$$(3.7) \quad [u, v]_g = \int_0^\infty \int_0^\infty \int_B G(s) u^*(x, t) v^*(x, s) ds dv,$$

for any $u, v \in X$.

From (2.1), (2.2), (2.5) and (2.8) we obtain

$$\begin{aligned}
 (3.8) \quad & \gamma * t_{ji,j} + f_i = \rho u_i, \\
 & \gamma * (\mu_{ji,j} + \varepsilon_{ijk} \tau_{jk}) + g_i = I_{ij} \varphi_j, \\
 & \gamma * (\pi_{ji,j} + \pi_i - \varphi_{,i} + E_i^0) = 0, \\
 & \gamma * (P_{i,i} - \varepsilon_0 \varphi_{,ii}) = \gamma * f, \text{ on } B \times I.
 \end{aligned}$$

To the field Eqs. (3.8) we adjoin the homogeneous boundary conditions.

We say that $U = (u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3, P_1, P_2, P_3, \varphi)$ is an admissible ten-dimensional vector field if $u_i, \varphi_i, P_i, \varphi \in X, i=1,2,3$.

By a solution of the problem we mean an admissible ten-dimensional vector field U that satisfies the Eqs. (3.8) and the homogeneous boundary conditions.

Let D be the set of all admissible ten-dimensional vector fields U that satisfy the homogeneous boundary conditions.

We introduce the operators $L_i (i=1,10)$ on D defined by

$$\begin{aligned}
 (3.9) \quad & L_i U = \rho u_i - \gamma * t_{ji,j} = f_i, \\
 & L_{i+3} U = I_{ij} \varphi_j - \gamma * (\mu_{ji,j} + \varepsilon_{ijk} \tau_{jk}) = g_i, \\
 & L_{i+6} U = \gamma * (-\pi_{ji,j} - \pi_i + \varphi_{,i}) = \gamma * E_i^0, \\
 & L_{10} U = \gamma * (P_{i,i} - \varepsilon_0 \varphi_{,ii}) = \gamma * f, \quad i=1,2,3.
 \end{aligned}$$

If we introduce the notations

$$LU = (L_1 U, L_2 U, \dots, L_{10} U), \quad F = (f_i, g_i, \gamma * E_i^0, \gamma * f),$$

then the Eqs. (3.8) can be written in the form of one vector equation

$$(3.10) \quad LU = F, \text{ on } B \times I.$$

Let $U = (u_i, \varphi_i, P_i, \varphi)$ and $V = (\bar{u}_i, \bar{\varphi}_i, \bar{P}_i, \bar{\varphi})$ be ten-dimensional vector fields on $\bar{B} \times I$ such that $u_i, \varphi_i, P_i, \varphi, u, u_i, \varphi_i, P_i, \varphi \in Y$.

We introduce the notation

$$(3.11) \quad [U, V]_g = \sum_{i=1}^3 \{ [u_i, \bar{u}_i]_g + [\varphi_i, \bar{\varphi}_i]_g + [P_i, \bar{P}_i]_g \} + [\varphi, \bar{\varphi}]_g.$$

In the linear theory the energy density of deformation and polarization is given by [2]

$$\begin{aligned}
 (3.12) \quad W = & \frac{1}{2} A_{ijkl} e_{ij} e_{kl} + A'_{ijkl} e_{ij} x_{kl} + A_{ijklm} e_{ij} x_{klm} - \\
 & - A_{ijk} e_{ij} P_k - G_{ijkl} e_{ij} P_{l,k} + \frac{1}{2} B_{ijkl} x_{ij} x_{kl} + B_{ijklm} x_{ij} x_{klm} - \\
 & - B_{ijk} x_{ij} P_k - G'_{ijkl} x_{ij} P_{l,k} + \frac{1}{2} D_{ijklmn} x_{ijk} x_{lmn} -
 \end{aligned}$$

$$-B_{ijkl}x_{ijk}P_l - G_{ijklm}x_{ijk}P_{m,l} - \frac{1}{2}A_{ij}P_iP_j - \\ -A_{ijk}P_iP_{k,j} - \frac{1}{2}D_{ijkl}P_{j,i}P_{l,k} + C_{ij}P_{j,i}.$$

The coefficients in (3.12) satisfy the symmetry relations [2], [7]

$$(3.13) \quad \begin{aligned} A_{ij} &= A_{ji}, \quad A_{ijk} = A_{kij}, \quad A_{ijk} = A_{kij}, \quad A_{ijkl} = A_{klij}, \\ A_{ijkl} &= A_{klij}, \quad A_{ijklm} = A_{klmij} = A_{ijlkm}, \quad B_{ijk} = B_{kij}, \\ B_{ijkl} &= B_{klij}, \quad B_{ijkl} = B_{jikl} = B_{lijk}, \quad B_{ijklm} = B_{ijlkm} = B_{klmij}, \\ D_{ijklmn} &= D_{lmnij} = D_{jiklmn}, \quad D_{ijkl} = D_{klij}, \quad G_{ijkl} = G_{klij}, \\ G_{ijkl} &= G_{klij}, \quad G_{ijklm} = G_{lmijk} = G_{jiklm}. \end{aligned}$$

If the boundary conditions (2.3) are homogeneous, then from (2.6) we obtain

$$(3.14) \quad \begin{aligned} \int_B \left[f_i^{(1)} * u_i^{(2)} + g_i^{(1)} * \varphi_i^{(2)} + \gamma * \left(E_i^{0(1)} * P_i^{(2)} + f^{(1)} * \varphi^{(2)} \right) \right] dv = \\ = \int_B \left[f_i^{(2)} * u_i^{(1)} + g_i^{(2)} * \varphi_i^{(1)} + \gamma * \left(E_i^{0(2)} * P_i^{(1)} + f^{(2)} * \varphi^{(1)} \right) \right] dv. \end{aligned}$$

Theorem 3.1. Assume that:

- i) the density field ρ is strictly positive;
- ii) the constitutive coefficients satisfy the symmetry relations (3.13);
- iii) the energy density of deformation and polarization (3.12) is a positive semi-definite quadratic form;
- iv) I_{ij} is positive definite tensor.

Then, the operator L is g -positive.

Proof. Consider two external data system $\Sigma^{(\alpha)}$ ($\alpha=1,2$) with null boundary data. Let $U^{(1)} = (u_i^{(1)}, \varphi_i^{(1)}, P_i^{(1)}, \varphi^{(1)})$ and $U^{(2)} = (u_i^{(2)}, \varphi_i^{(2)}, P_i^{(2)}, \varphi^{(2)})$ be solutions corresponding to $\Sigma^{(1)}$ and $\Sigma^{(2)}$ respectively. Then

$$(3.15) \quad LU^{(1)} = F^{(1)}, \quad LU^{(2)} = F^{(2)}.$$

If we take the Laplace transform of (3.14), by means of the relation (3.7), it results

$$(3.16) \quad [LU^{(1)}, U^{(2)}]_g = [U^{(1)}, LU^{(2)}]_g,$$

for any $U^{(1)}, U^{(2)} \in D$.

From (2.7) we obtain

$$\begin{aligned}
 & \int_B \gamma * [\tau_{ij} * e_{ij} + \mu_{ij} * \kappa_{ij} + \mu_{ijk} * \kappa_{ijk} - \pi_i * P_i] dv + \\
 & + \int_B \gamma * [\varepsilon_0 E_i * E_i + (\pi_{ij} - C_{ij}) * P_{j,i}] dv = \\
 (3.17) \quad & = \int_B \left[f_i * u_i + g_i * \varphi_i + (\gamma * E_i^0) * P_i + (\gamma * f) * \varphi \right] dv - \\
 & - \int_B (\rho u_i * u_i + I_{ij} \varphi_i * \varphi_j) dv .
 \end{aligned}$$

If we take Laplace transform of this relation by means of the formulae (3.1) and (3.7) we get

$$\begin{aligned}
 [LU, U]_g &= \int_0^\infty \int_B G(s) (\rho u_i^* u_i^* + I_{ij} \varphi_i^* \varphi_j^*) ds dv + \int_0^\infty \int_B 2G(s) \gamma^* W^* ds dv = \\
 (3.18) \quad & = \int_0^\infty \int_0^\infty \int_B g(t+s) [\rho u_i(t) u_i(s) + I_{ij} \varphi_i(t) \varphi_j(s)] dt ds dv + \\
 & + \int_0^\infty \int_0^\infty \int_B 2g(t+s) \gamma(t) W(s) dt ds dv .
 \end{aligned}$$

Using the hypotheses i) - iv), the relations (3.16) and (3.18) we obtain the desired results.

Let Φ_g be the functional on D defined by

$$(3.19) \quad \Phi_g\{U\} = [LU, U]_g - 2[U, F]_g .$$

In view of Lemma 3.1 and Theorem 3.1 we are led to the

Theorem 3.2. Suppose that the density field is strictly positive, I_{ij} is a positive definite tensor and the energy density of deformation and polarization is a positive semi-definite quadratic form. Let Φ_g be the functional defined by (3.19).

Further, let $U \in D$ be a solution of the mixed problem with null boundary data. Then

$$\Phi_g\{U\} \leq \Phi_g\{U^0\} ,$$

for every $U^0 \in D$.

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PRINCIPIU DE MINIM PENTRU TEORIA PIEZOELECTRICITĂȚII MATERIALELOR MICROPOLARE NESIMPLE CU GRADIENT AL POLARIZĂRII

(Rezumat)

Se prezintă un principiu de minim pentru teoria liniară a piezoelectricității materialelor micropolare nesimple, arătând că soluția problemei mixte cu condiții pe frontieră nule minimizează o anumită funcțională.

EXACT SOLUTIONS OF EINSTEIN'S EQUATIONS ON $\mathbb{R} \times S^3$ WITH DYNAMICAL METRICS

BY

GH. ZET, C. DARIESCU and AURA DARIESCU

1. Introduction. The space-time $\mathbb{R} \times S^3$ has recently known a growing interest in physics. Different authors have developed field theories on $\mathbb{R} \times S^3$ topology [1]...[8]. The explanation is that the surface of the sphere S^3 is a space with considerably more symmetry than other spaces. It is convenient to leave the temporal dimension continuous and flat, so the theory can be canonically quantized.

In this paper we obtain two classes of exact solutions for Einstein's equations of the gravitational field associated to the manifold $\mathbb{R} \times S^3$. First the geometry of the space-time $\mathbb{R} \times S^3$ is reviewed. Then, following an idea of Sen [7], we consider the case when the radius ρ of S^3 is time-dependent, i.e. $\rho = f(t)$. The Einstein field equations are written and two classes of exact solutions are obtained. The first class corresponds to a static metric on $\mathbb{R} \times S^3$, when $\rho = f(t) = \text{const.}$, and the second one to a dynamic metric. In the static case we show that the cosmological constant Λ must be introduced in the model, being interpreted as a universal compensation pressure. When the metric is dynamized we obtain exact solutions describing a pure, homogeneous, isotropic and non-polarized electromagnetic radiation. The cosmological constant Λ is not necessary in this case.

Finally, the principal results of the paper are summarized and some observations about a gravitation theory on the space-time $\mathbb{R} \times S^3$ are given.

2. The Geometry of $\mathbb{R} \times S^3$. The metric used in this space-time is of time-like type. The sphere S^3 can be parametrized in terms of either Cartesian coordinates x^k ($k=1, 2, 3$) or angles χ, θ, φ . If the radius of S^3 is denoted by ρ then we can write

$$(2.1) \quad (x^1)^2 + (x^2)^2 + (x^3)^2 + u^2 = \rho^2, \quad u = x^4,$$

where

$$(2.2) \quad \begin{aligned} x^1 &= \rho \sin \chi \sin \theta \cos \varphi, & x^2 &= \rho \sin \chi \sin \theta \sin \varphi, \\ x^3 &= \rho \sin \chi \cos \theta, & x^4 &= \rho \cos \chi. \end{aligned}$$

This parametrization is equivalent to those given in [7] and [8].

The metric of the sphere S^3 can be written under the form

$$(2.3) \quad d\sigma^2 = \delta_{\alpha\beta} d\chi^\alpha d\chi^\beta + du^2, \quad \alpha, \beta = 1, 2, 3.$$

With the parametrization (2.2) we obtain

$$(2.4) \quad d\sigma^2 = \rho^2 [(d\chi)^2 + \sin^2 \chi (d\theta)^2 + \sin^2 \chi \sin^2 \theta (d\varphi)^2].$$

Following an idea of D. Sen [7] we will suppose that the radius ρ is time-dependent, i.e. we put $\rho = f(t)$, where $f(t)$ is a function dependent on "the epoch" t . Then, the metric of the space-time $\mathbb{R} \times S^3$ can be written as

$$(2.5) \quad ds^2 = f^2(t) [(d\chi)^2 + \sin^2 \chi (d\theta)^2 + \sin^2 \chi \sin^2 \theta (d\varphi)^2] - dt^2.$$

Now, it is convenient to use a dual orthonormalized rigid basis $\{\omega^a\}$ ($a = 1, 2, 3, 4$) of Cartan type, defined as

$$(2.6) \quad ds^2 = \eta_{ab} \omega^a \omega^b, \quad \eta_{ab} = \text{diag}[1, 1, 1, -1].$$

Then, we have

$$(2.7) \quad \omega^1 = f d\chi, \quad \omega^2 = f \sin \chi d\theta, \quad \omega^3 = f \sin \chi \sin \theta d\varphi, \quad \omega^4 = dt.$$

The corresponding basis vectors $\{e_a\}$ ($a = 1, 2, 3, 4$) are

$$(2.8) \quad e_1 = \frac{1}{f} \frac{\partial}{\partial \chi}, \quad e_2 = \frac{1}{f \sin \chi} \frac{\partial}{\partial \theta}, \quad e_3 = \frac{1}{f \sin \chi \sin \theta} \frac{\partial}{\partial \varphi}, \quad e_4 = \frac{\partial}{\partial t}.$$

The Cartan solutions of structure have the form [9]

$$(2.9) \quad d\omega^a = g^{ad} \Gamma_{dbc} \omega^b \wedge \omega^c, \quad \Gamma_{ab} = \Gamma_{abc} \omega^c,$$

$$(2.10) \quad d\Gamma_{ab} + g^{cd} \Gamma_{ac} \wedge \Gamma_{db} = \frac{1}{2} R_{abef} \omega^e \wedge \omega^f,$$

where the latin indices $a, b, c, \dots$ take the values 1, 2, 3, 4. In this case

$$g^{ab} = \eta^{ab}.$$

Using (2.7) and (2.9) we obtain

$$(2.11)_a \quad \Gamma_{1[bc]} \omega^b \wedge \omega^c = -\frac{f'}{f} \omega^1 \wedge \omega^4,$$

$$(2.11)_b \quad \Gamma_{2[bc]} \omega^b \wedge \omega^c = \frac{\cotan \chi}{f} \omega^1 \wedge \omega^2 - \frac{f'}{f} \omega^2 \wedge \omega^4,$$

$$(2.11)_c \quad \Gamma_{3[bc]} \omega^b \wedge \omega^c = \frac{\cotan \chi}{f} \omega^1 \wedge \omega^3 + \frac{\cotan \theta}{f \sin \chi} \omega^2 \wedge \omega^3 - \frac{f'}{f} \omega^3 \wedge \omega^4,$$

$$(2.11)_d \quad \Gamma_{4[bc]} \omega^b \wedge \omega^c = 0,$$

where $b < c$ and

$$(2.15) \quad \Gamma_{a[bc]} = \Gamma_{abc} - \Gamma_{acb}.$$

We noted by $f'(t)$ the derivative of $f(t)$ with respect to the "epoch" t .

By identification of coefficients of the two-forms $\omega^a \wedge \omega^b$ in Eqs. (2.11) we obtain the following non-null coefficients of connection:

$$\begin{aligned}
 (2.16) \quad & \Gamma_{141} = -\Gamma_{411} = \frac{f'}{f}, \quad \Gamma_{212} = -\Gamma_{122} = \frac{\cotan \chi}{f}, \quad \Gamma_{242} = -\Gamma_{422} = \frac{f'}{f}, \\
 & \Gamma_{313} = -\Gamma_{133} = \frac{\cotan \chi}{f}, \quad \Gamma_{323} = -\Gamma_{233} = \frac{\cotan \theta}{f \sin \chi}, \quad \Gamma_{343} = -\Gamma_{433} = \frac{f'}{f}.
 \end{aligned}$$

Then, the 1-forms of connection Γ_{ab} in (2.9) can be written as

$$\begin{aligned}
 (2.17) \quad & \Gamma_{12} = -\frac{\cotan \chi}{f} \omega^2, \quad \Gamma_{13} = -\frac{\cotan \chi}{f} \omega^3, \quad \Gamma_{14} = \frac{f'}{f} \omega^1, \\
 & \Gamma_{23} = -\frac{\cotan \theta}{f \sin \theta} \omega^3, \quad \Gamma_{24} = \frac{f'}{f} \omega^2.
 \end{aligned}$$

Now, introducing (2.17) in (2.10) we have

$$(2.18)_a \quad R_{\alpha\beta} e^{\alpha} \omega^{\beta} \wedge \omega^{\gamma} = \frac{1}{f^2} [1 + (f')^2] \omega^{\alpha} \wedge \omega^{\beta},$$

$$(2.18)_b \quad R_{\alpha 4} e^{\alpha} \omega^{\beta} \wedge \omega^{\gamma} = -\frac{f''}{f} \omega^{\alpha} \wedge \omega^{\gamma}, \quad \alpha = 1, 2, 3, \quad e < f.$$

The Eqs. (2.18) show that the non-null components of the Riemann curvature tensor are given by

$$(2.19) \quad R_{1212} = R_{1313} = R_{2323} = \frac{1}{f^2} [1 + (f')^2],$$

$$R_{1414} = R_{2424} = R_{3434} = -\frac{f''}{f}.$$

The Ricci tensor R_{ab} , will have the following non-null components:

$$(2.20)_a \quad R_{ab} = R_{acb}^c = \eta^{cd} R_{cadb},$$

$$(2.20)_b \quad R_{11} = R_{22} = R_{33} = \frac{f''}{f} + \frac{2}{f^2} [1 + (f')^2], \quad R_{44} = -\frac{3f''}{f}.$$

Then, the Ricci scalar R is given by the expression

$$(2.21) \quad R = 6 \left[\frac{f''}{f} + \frac{1}{f^2} (1 + (f')^2) \right]$$

We can see that for $f(t) = \rho = 1$ we have $R = 6$, a result which is well known from geometry.

3. Einstein's Equations. We calculate the components G_{ab} of the Einstein tensor by

$$(3.1) \quad G_{ab} = R_{ab} - \frac{1}{2} \eta_{ab} R.$$

Introducing (2.20) and (2.21) in (3.1) we have

$$(3.2) \quad \begin{aligned} G_{11} &= G_{22} = G_{33} = - \left[\frac{2f''}{f} + \frac{1}{f^2} (1 + (f')^2) \right], \\ G_{44} &= \frac{3}{f^2} [1 + (f')^2]. \end{aligned}$$

The gravitational field equations will be assumed to be given by the Einstein field equations

$$(3.3) \quad G_{ab} = R_{ab} - \frac{1}{2} \eta_{ab} R = k T_{ab},$$

where T_{ab} is the energy-momentum tensor and k is the Newton constant. By comparing (3.3) and (3.2) we obtain the following expressions for the components of T_{ab} :

$$(3.4) \quad \begin{aligned} T_{\alpha\beta} &= -\frac{1}{k} \delta_{\alpha\beta} \left[\frac{2f''}{f} + \frac{1}{f^2} (1 + (f')^2) \right], \\ T_{44} &= \frac{3/k}{f^2} [1 + (f')^2]. \end{aligned}$$

We will find solutions of the Eqs. (3.3) for the following two cases of interest in physics:

i) *The case when $f(t) = \rho = \text{const.}$, i.e. when we have a static metric in the space-time $\mathbb{R} \times S^3$.* Then, from Eqs. (3.4) we obtain

$$(3.5) \quad T_{\alpha\beta} = -\frac{1/k}{\rho^2} \delta_{\alpha\beta}, \quad T_{44} = \frac{3/k}{\rho^2}.$$

The first Eq. (3.5) shows that there are only negative principal pressures. On the other hand from the second Eq. (3.5) it results that the density of energy is well defined, i.e. it is a positive quantity. But, we can see that in this case it is necessary to introduce the cosmological constant Λ . The constant Λ may be interpreted as a universal pressure which has a compensation role within the model of a perfect homogeneous and isotropic fluid. The matter energy-momentum tensor of the fluid [9] is

$$(3.6) \quad T_{ab} = (w + p) u_a u_b + \left[p - \frac{\Lambda}{k} \right] g_{ab}.$$

If we suppose that the fluid is static, i.e.

$$(3.7) \quad u_\alpha = 0, \quad \alpha = 1, 2, 3,$$

then from Eq. (3.5) we obtain the following expressions for the pressure p and the density of energy w :

$$(3.8) \quad p = \frac{1}{k} \left[\Lambda - \frac{1}{\rho^2} \right], \quad w = \frac{1}{k} \left[\frac{3}{\rho^2} - \Lambda \right].$$

To obtain positive values for p and w , we must impose the following constraint for the curvature radius of S^3 ,

$$(3.9) \quad 1 \leq \Lambda \rho^2 \leq 3.$$

ii) *The case when there is a dynamic metric on $\mathbb{R} \times S^3$.* In this case we can obtain an exact solution of the Einstein field Eqs. (3.3) if we consider T_{ab} as the

tensor energy-momentum of an electromagnetic field of pure radiation. This is important for the quantization of the gravitational and electromagnetic fields on $\mathbb{R} \times S^3$ topology.

If we compare the second expression (3.5) of the component T_{44} with the well known formula of the energy density [9]

$$(3.10) \quad w = 3p,$$

we can write the tensor energy-momentum T_{ab} under the form [9]

$$(3.11) \quad T_{ab} = \frac{w}{3} (4\delta_a^4 \delta_b^4 + \eta_{ab}).$$

Introducing (3.11) in (3.5) we obtain

$$(3.12) \quad \frac{2}{f} f'' + \frac{1}{f^2} [1 + (f')^2] = -\frac{k}{3} w,$$

$$(3.13) \quad \frac{1}{f^2} [1 + (f')^2] = \frac{k}{3} w.$$

From the Eqs. (3.12) and (3.13) it results

$$(3.14) \quad f f'' + (f')^2 + 1 = 0.$$

The solution of Eq. (3.14) is

$$(3.15) \quad f(t) = [a^2 - (b \pm t)^2]^{1/2},$$

where a and b are two integration real constants. The density of energy w for the electromagnetic field of pure radiation can be obtained by introducing the Eq. (3.15) in (3.13):

$$(3.16) \quad w = \frac{3a^2/k}{[a^2 - (b \pm t)^2]^2}.$$

It is important to observe that the dynamization of the metric (2.5) by the function $f(t)$ makes inadequate the introduction of the cosmological constant Λ in the model, at least in the case of a pure radiation.

4. Conclusions. We have studied the exact solutions of the Einstein field equations on $\mathbb{R} \times S^3$ topology. There are two cases when these equations admit exact solutions:

i) *The metric of $\mathbb{R} \times S^3$ is static.* In this case it is necessary the existence of a cosmological pressure Λ/k satisfying the constraint (3.9). The corresponding exact solution describes then a static perfect fluid, homogeneous and isotropic.

ii) *The metric of $\mathbb{R} \times S^3$ is dynamized by a function $f(t)$, as in Eq. (2.5).* In this case the Einstein field equations impose the expression (3.15) for the function $f(t)$. This solution corresponds to an electromagnetic field of pure radiation with the density of energy w given by Eq. (3.16). The cosmological constant Λ is not necessary in this case.

The results of this paper can be generalized to an arbitrary linear connection on $\mathbb{R} \times S^3$. We obtain then a theory of gravitation on $\mathbb{R} \times S^3$ topology. Such a theory is given in [10] where a comparison with other results is also made. A gauge theory of

gravitation on $\mathbb{R} \times S^3$ topology will be done in a forthcoming paper [11].

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SOLUȚII EXACTE ALE ECUAȚIILOR LUI EINSTEIN ÎN $\mathbb{R} \times S^3$ CU METRICI DINAMICE

(Rezumat)

Se consideră spațiul-timp $\mathbb{R} \times S^3$ ca varietate pseudo-riemanniană cu metrica (2.5) și se obțin două clase de soluții exacte pentru ecuațiile Einstein ale câmpului gravitațional asociat cu varietatea $\mathbb{R} \times S^3$. Prima clasă de soluții corespunde unei metrici statice pe $\mathbb{R} \times S^3$, când constanta cosmologică Λ trebuie introdusă în model. Cea de a doua clasă de soluții corespunde unei metrici dinamice, descriind o radiație electromagnetică pură, omogenă, izotropă și nepolarizată, care nu mai necesită prezența constantei Λ .

ABOUT THE DEGENERATION OF THE MOVEMENT OF THE SIMPLE PENDULUM

BY

MARICEL AGOP, IULIAN ARCHIP, I. A. CIUREANU, V. GRIGA

I. BRÎNZĂ and M. CAIA

1. Introduction. The fact that for a class of nonlinear oscillators, in particular for the pendulum, the close solution can be formulate using the elliptic functions [1]...[3] is well known. As a rule, such a presentation is formal because an exactly physical meaning is not assigned to the period of this function specially to that imaginary. Consequently, we discuss the solutions for the equations of the pendulum according to the behavior of the elliptical function periods. So, we show that the solitonic movement [4] is found in the degeneration of the Jacobi elliptical functions in the asymptotic limit of the imaginary period and the oscillatory and circular motions as degeneration of the same functions in the asymptotic limit of the real period.

The fact that the classical movements of the Simple Pendulum are found through the limit in the periods of a certain elliptic function, periods with a direct physical meaning, justifies the introduction of the binary quantities in the study of non-linear dynamics.

2. The Close Solution for the Equation of the Simple Pendulum. The equation raised for discussion is

$$(2.1) \quad \ddot{\varphi} + \omega_0^2 \sin \varphi = 0,$$

where φ is the angular displacement, $\omega_0 = 1$ (with a suitable scalation) and $(\ddot{}) = d^2/dt^2$. Following a known proceeding [5] for conservative system we find for the period of the motion

$$(2.2) \quad T = \oint \frac{\partial \varphi}{\partial h / \partial p},$$

where $h = \frac{1}{2}p^2 + \omega_0^2 \cos \varphi$ is the Hamilton function for the simple pendulum and p is

the generalized momentum.

Using the substitution $\sin \frac{1}{2} \varphi = z$ we have

$$(2.3) \quad T = 2 \left(\dot{\varphi}_0^2 + 4\omega_0^2 \sin^2 \frac{1}{2} \varphi_0 \right)^{-\frac{1}{2}} \int_0^{\pi/2} \frac{dz}{[(1-z^2)(1-k^2 z^2)]^{1/2}}, \quad \varphi_0 = \varphi|_{t=t_0};$$

so T is expressed through an elliptic integral with Lagrange normal shape, the modulus being

$$(2.4) \quad k^2 = \left[\left(\frac{\dot{\varphi}^2}{2\omega_0^2} \right)^2 + \sin^2 \frac{1}{2} \varphi_0 \right]^{-1}.$$

The inverse of the integral [3] is the Jacobi elliptical function

$$(2.5) \quad z = \operatorname{sn} \left[\frac{1}{2} \left(\dot{\varphi}_0^2 + 4\omega_0^2 \sin^2 \frac{1}{2} \varphi_0 \right)^{\frac{1}{2}} T \right],$$

with the real period

$$(2.6) \quad T_1 = 4 \left(\dot{\varphi}_0^2 + 4\omega_0^2 \sin^2 \frac{1}{2} \varphi_0 \right)^{-\frac{1}{2}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$$

and the imaginary period

$$(2.7) \quad T_2 = 2i \left(\dot{\varphi}_0^2 + 4\omega_0^2 \sin^2 \frac{1}{2} \varphi_0 \right)^{-\frac{1}{2}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k'^2 \sin^2 \varphi}},$$

where

$$(2.8) \quad k^2 + k'^2 = 1.$$

3. Circular Motion. If $\varphi_0 \equiv 0$ this implies $k^2 \rightarrow 0$ or equivalently $\dot{\varphi}_0^2 / 2\omega_0^2 \rightarrow \infty$ that leads to $T_1 \rightarrow 2\pi / \dot{\varphi}_0$, $T_2 \rightarrow \infty$.

In this condition $\operatorname{sn} \rightarrow \sin$ and

$$(3.1) \quad \varphi = \dot{\varphi}_0 (t - t_0).$$

We obtain that the known equation for the circular motion is found as the result of the degeneration of the chosen elliptic function in the imaginary period.

4. Solitonic Motion. For $\varphi_0 \equiv 0$ and $k^2 \rightarrow 1$ we obtain from (2.3), (2.6) and

$$(2.7) \quad T_1 \rightarrow \infty, \quad T_2 = \frac{i\pi}{\dot{\varphi}_0} \quad \text{and} \quad \varphi = 4 \tan^{-1} e^{\omega(t-t_0)},$$

that leads for speed to a solitonic solution.

This motion is allowed on the separatrix [3] and that is a direct result of the degeneration of the elliptic function in real period.

5. Oscillatory Motion. In the allowed region for the oscillatory motion with the substitution

$$(5.1) \quad k_0^2 = \left[\frac{\dot{\varphi}_0}{2\omega_0} \right]^2 + \sin^2 \frac{1}{2} \psi_0 \quad \text{and} \quad \sin \frac{1}{2} \varphi = k_0 u,$$

the integral (2.2) becomes

$$(5.2) \quad T = \frac{1}{\omega_0} \oint \frac{du}{\sqrt{(1-u^2)(1-k_0^2 u^2)}},$$

with the inverse

$$(5.3) \quad u = \operatorname{sn}[\omega(t-t_0)]$$

as the solution for the equation [1].

The real respectively imaginary periods are

$$(5.4) \quad T_1 = \frac{4}{\omega_0} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k_0^2 \sin^2 \alpha}}, \quad T_2 = \frac{2i}{\omega_0} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k_0^2 \sin^2 \alpha}}$$

satisfying condition (2.8).

We make the restriction $\varphi_0 \equiv 0$. If $k_0^2 \rightarrow 0$ then $T_1 \rightarrow T_0 = 2\pi/\omega_0$ (isochronic motion) and $T_2 \rightarrow \infty$; we find the linear oscillations period as an asymptotic limit in the imaginary period. The isochronic oscillations period is quite the considered elliptic function real period.

6. Conclusions. The above presented results argue the introduction of the "binary quantities" in the study of non-linear oscillations. According with this concept, any physical measurable quantity must be defined as a complex function in which the real part points to the average of the respective quantity and the complex part to the dispersion. In the discussed case the deterministic motions are separated by the soliton state on which the motion can be fluctuating.

About the considerations connected to the dispersion introduced by the imaginary period and the damping of the chaotic motions in presence of an external excitation we will return in a future work.

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ASUPRA DEGENERĂRII MIȘCĂRII PENDULULUI SIMPLU

(Rezumat)

Utilizând funcțiile eliptice se arată că mișcarea periodică a pendulului simplu se obține ca degenerare a funcției eliptice în perioada imaginară, în timp ce mișcarea solitonă rezultă ca degenerare a aceleiași funcții eliptice în perioada reală.

În acest fel se pot introduce în studiul fenomenelor fizice mărimile binare.

ON NON-ISOTHERMAL CHARGE DECAY FOR ELECTRETS
OBTAINED BY CHARGE INJECTION

BY

EUGEN NEAGU

1. Introduction. The theory of excess charge discharge in dielectric media is a difficult problem and was considered by many authors [1]...[3]. The parameters being time and space-dependent it is necessary to solve nonlinear partial differential equations, for which no general solution could be driven. We try to solve this problem on the basis of some hypotheses which result from physical considerations.

2. Theory. For a given space charge density in the sample there is a plane for which the electric field is equal to zero [4]. This plane is the zero field plane. From this plane two currents moves towards the shorted electrodes. Generally, the current density is given by

$$(1) \quad j(x, T) = \varepsilon \frac{\partial E(x, T)}{\partial t} + q \mu_n n_e(T) E(x, T) + q \mu_n n(x, T) E(x, T) + j_{\text{diff}},$$

where ε is the dielectric constant of the sample, q — the electric charge, $\mu_n(T)$ — the mobility for negative charges, $n_e(T)$ — the space-independent equilibrium carriers density and $n(x, T)$ — the space-dependent excess carriers density which constitute the electric charge; j_{diff} is the diffusion current and will be neglected.

For plane parallel geometry, we consider that the space distribution of carriers density is given by

$$(2) \quad n(x, T) = \begin{cases} n(T) & b(T) \leq x \leq a(T), \\ N_0 & B \leq x \leq A, \\ 0 & 0 < x < b(T), \\ 0 & a(T) < x < L, \end{cases}$$

From Poisson equation for electric field in the dielectric results

$$\begin{aligned}
 x \in [0, b(T)], \quad E(0, T) &= \frac{qn(T)}{\varepsilon} [a_0(T) - b(T)], \\
 (3) \quad x \in [a(T), L], \quad E(L, T) &= -\frac{qn(T)}{\varepsilon} [a(T) - a_0(T)], \\
 x \in [b(T), a(T)], \quad E(x, T) &= \frac{qn(T)}{\varepsilon} [a_0(T) - x],
 \end{aligned}$$

where $E[a_0(T), T]=0$ gives the position of the zero field plane.

We assume that the charge distribution remains uniform for $T > T_0$. To calculate the temperature variation of the electric field we assume that for the temperature modification of charge cloud edges it is possible to use the following simple expressions:

$$\begin{aligned}
 (4) \quad a(T) &= A + \frac{k}{s}(T - T_0), \\
 b(T) &= B - \frac{k}{s}(T - T_0),
 \end{aligned}$$

where s is the heating rate and k — a constant parameter. We will consider two temperature intervals: a first for $T < T_1$, where T_1 results from the condition $b(T_1)=0$ and a second for $T > T_1$. For the first temperature interval from Eqs. (4) and charge conservation law results

$$(5) \quad n(T) = \frac{N_0(A-B)}{A-B+(k/s)(T-T_0)}.$$

The current released in external circuit at $x=0$ for the first temperature interval can be obtained by integration of Eq. (4) from $x=0$ to $x=a_0$:

$$\begin{aligned}
 (6) \quad j_{11}(T) &= \frac{1}{a_0(T)} \left\{ \varepsilon s \frac{\partial}{\partial T} \int_0^{a_0} E(x, T) dx + q \mu_n(T) n_e(T) \int_0^{a_0} E(x, T) dx \right\} = \\
 &= \frac{q}{2a_0} s \frac{\partial}{\partial T} \left[n(T) (a_0^2 - b_0^2(T)) \right] + \frac{q^2 \mu_n(T) n_e(T) n(T)}{2\varepsilon a_0} [a_0^2 - b^2(T)].
 \end{aligned}$$

From (4), (5) and (6) results

$$(7) \quad j_{11} = \frac{qN_0(A-B)}{4a_0} \left\{ \frac{\sigma(T)}{\varepsilon} \left[a_0 + B - \frac{k}{s}(T - T_0) - k \right] \right\}.$$

The released current is positive, which means that it flows in opposite direction to the homopolar charging current. The current is zero if $N_0=0$ or if $A-B=0$. It increases with the temperature as $\sigma(T)$. This fact is very important because the initial rise method [5] can be used for the determination of activation energy. For the same temperature interval the current released in the external circuit at $x=L$ can be obtained in the same way starting from relation (1):

$$(8) \quad j_{21} = \frac{qN_0(A-B)}{4(L-a_0)} \left\{ k - \frac{\sigma(T)}{\varepsilon} \left[2L - A - a_0 - \frac{k}{s}(T-T_0) \right] \right\}.$$

From (8) it results that the current is negative and obviously not equal with $j_{11}(T)$. It is zero if $N_0=0$ or $A-B=0$.

The currents released in external circuit will be equal if

$$j_{11} + j_{21} = 0.$$

If we consider this relation for $T_1 = T_0 + s/k$ results $a_0 = L/2$. The released currents in external circuit will be equal if zero field plane is in the middle of the sample, that is for a symmetrical charge distribution in the sample.

We have proposed to improve the Thermally Stimulated Discharge Current Method [6], [7] by using two electrometers to measure the released currents in external circuit from both faces of the sample. Experimentally we have found that the two currents are not equal. The relations (7) and (8) show now clearly that the two currents are not equal.

For $T > T_1$ the charge cloud reaches the electrode at $x=0$. The zero field plane start to move and we have

$$(9) \quad a_0(T) = a(T) - \frac{k\varepsilon}{\sigma(T)}.$$

From the continuity equation we obtain

$$(10) \quad n(T) = N_0 \exp \left[-\frac{1}{s} \varepsilon \int_{T_0}^T \sigma(T) dT \right].$$

For this case the current released in external circuit at $x=0$ is

$$(11) \quad j_{12}(T) = qkn(T) + \frac{sn(T)k\varepsilon qW}{k_B T^2} \frac{1}{\sigma(T)},$$

where k_B is Boltzmann's constant. The current is positive and decreases if the temperature increases. For the current released in external circuit at $x=L$, for the same temperature interval results

$$(12) \quad j_{22}(T) = -\frac{qn(T)k\varepsilon}{L-a_0} \frac{1}{\sigma(T)} \left\{ s \frac{\partial a_0(T)}{\partial T} + \frac{sW}{k_B T^2} \left[L - a_0(T) - \frac{k\varepsilon}{\sigma(T)} \right] \right\}.$$

The current is negative and depends on $a_0(T)$. From (11) and (12) it results that the two currents are not equal. From relation (12) it is possible to obtain information about the movement of the zero field plane.

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ASUPRA ATENUĂRII SARCINII NEIZOTERME LA ELECTREȚII OBTINUȚI PRIN INECȚIE DE SARCINĂ

(Rezumat)

Ținând seama că teoria descărcărilor sarcinilor în exces în medii dielectrice este foarte dificilă, conducând la ecuații cu derivate parțiale neliniare, se prezintă o soluție a problemei pe baza unor ipoteze simplificatoare ce rezultă din considerații fizice directe.

PHOTOANODES OF SINTERED $\alpha\text{-Fe}_2\text{O}_3$

BY

SERGIU ISTRATE and EUGEN NEAGU

1. Introduction. Although the free energy required for the decomposition of water into gaseous H_2 and O_2 is only 1.23 eV, while the peak of the solar spectrum occurs at a photon energy of about 2.4 eV, solar energy cannot be utilized for the production of H_2 by direct photodecomposition of water, because the threshold energy for this direct reaction is about 6.5 eV.

Fujishima and Honda [1] have shown, however, that this threshold can be greatly reduced if decomposition is accomplished by means of photoelectrolysis, a process in which a semiconductor is used as a catalyst. These authors immersed Pt and single-crystal TiO_2 electrodes in an aqueous electrolyte and connected them externally to form an electrolytic cell. When the semiconducting n -type TiO_2 was illuminated, current flowed in external circuit, and they reported the liberation of O_2 and H_2 , respectively, at the TiO_2 and Pt electrodes, provided that the photon energy exceeded 3.0 eV, the energy gap of TiO_2 . Other materials have been studied [2] in a search for a stable electrode with a lower energy gap and appropriate electron affinity which would allow for efficient electrolysis of water. A promising electrode candidate is Fe_2O_3 which has an energy gap of 2.2 eV. This material was first investigated by Hardees and Bard [2] using chemically vapor deposited (CVD) films of Fe_2O_3 . Fe_2O_3 is a good electrode candidate since approximately 29% of the energy in air mass 1 (AM1) solar spectrum occurs for photon energies greater than its band gap of 2.2 eV. Correspondingly, 12% and 4% of solar spectrum exists at energies greater than the band gap energies of 2.7 and 3.1 eV for WO_3 [3] and TiO_2 respectively. Other stable electrode materials have larger band gaps: BaTiO_3 ($E_g = 3.3$ eV), SrTiO_3 ($E_g = 3.4$ eV) and SnO_2 ($E_g = 3.5$ eV). Common semiconductor with smaller band gap energies undergo dissolution of the electrode with conditions appropriate for photolysis. The results of Hardee and Bard [2] indicate $\alpha\text{-Fe}_2\text{O}_3$ is stable at anodic potentials.

In this paper we study the behaviour of $\alpha\text{-Fe}_2\text{O}_3$ impurified with ZrO_2 , TiO_2 and Nb_2O_5 , respectively, as semiconductor photoanode.

2. Experimental Procedure. The polycrystalline anodes were obtained by sintering $\alpha\text{-Fe}_2\text{O}_3$ and $\alpha\text{-Fe}_2\text{O}_3$ impurified powder. The oxide powder $\alpha\text{-Fe}_2\text{O}_3$, ZrO_2 , TiO_2 and Nb_2O_5 was homogenized, pressed and sintered at 1200°C during 24 hours; the shape of the anodes has been the pellet. The level of impurity was 1%; 0.1% and 0.01% respectively with ZrO_2 , TiO_2 and Nb_2O_5 respectively, in $\alpha\text{-Fe}_2\text{O}_3$.

The potential-current curves were obtained with the aid of the electrolysis cell; the *n*-type semiconductor ($\alpha\text{-Fe}_2\text{O}_3$ impurified or not) is immersed in aqueous electrolyte (NaOH , KOH or H_2SO_4) with a suitable counter electrode (platinum wire). Illumination of the semiconductor with photon energies greater than the band gap energy produces electron-hole pairs near the surface. Charge separation can occur in the presence of the electric field of a space charge depletion region at the surface. With upward band bending, holes move to the surface and take part in electrochemical oxygen production or other oxidation reactions often including dissolution of the electrode. Electron flow leads to reduction at the platinum counter electrode, which, under proper conditions results in hydrogen evolution. The semiconductor electrode (work electrode), counter electrode (platinum wire) and reference electrode (saturated calomel electrode — SCE) was introduced in the electrolyte. The voltage between SCE and work electrode was measured with a digital high impedance input voltmeter. The work electrode was illuminated by a 200 Watt vapour mercury lamp.

3. Results and Discussion. We determined the photocurrent versus applied potential. By photocurrent we understand the difference between the light current (the current when the work electrode is illuminated) and the dark current.

In the case of the pure $\alpha\text{-Fe}_2\text{O}_3$ anodes, it was not observed any current because the electrical resistance of this conductor is too large ($> 10\text{ M}\Omega$), and then it is necessary to impurify the ferric oxide. The samples with 0.01% impurity (ZrO_2 , TiO_2 and Nb_2O_5 respectively) have also a large electrical resistance. For the samples with 1% impurity we observed a very small photoresponse; we explain this observation by an inefficiency space charge depletion region. The samples impurified with 0.1% give a notable photoresponse only after the grinding from the face is exposed to radiation. We observed that during the cooling of the samples in the furnace, the surface probe reacts with oxygen, and yields a very high resistive superficial layer. We measured the electrical resistance before and after grinding, respectively, which modifies by two or three magnitude orders.

In Fig. 1 we represented the photocurrent versus external potential, applied between the anode from $\alpha\text{-Fe}_2\text{O}_3$ impurified with 0.1% Nb_2O_5 and the counter electrode (platinum wire cathode). The first curve 1 is obtained by the first exposition of the semiconductor surface to light. We observed that for 1 V external applied voltage, O_2 to the anode and H_2 to the cathode, respectively, are generated. In [4] it is affirmed that the electrolysis appears for 1.48 V external voltage.

We degreased the face of the sample and plotted the curve 2 and after 5 days the curve 3. The comparison of the 2 and 3 curves shows that the results are reproducible and the photoelectrode stability was good. The potential at which we

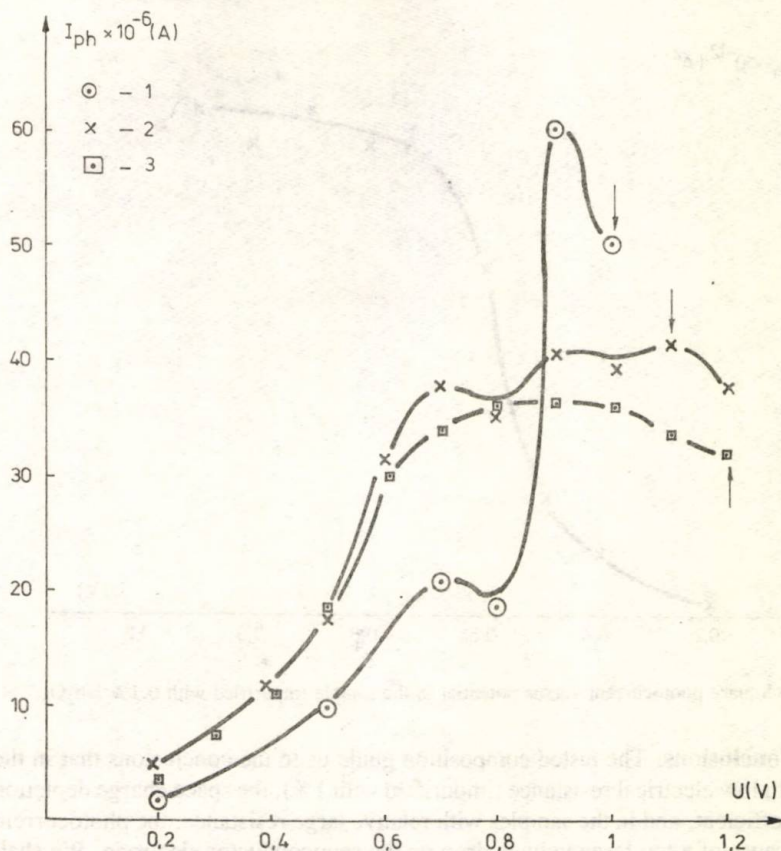


Fig. 1 — The potential-photocurrent curves in the sample impurified with 0.1% Nb_2O_5 .

observe the engendering of the O_2 to anode and H_2 to cathode, respectively, are marked by little arrows in the figure. Another conclusion could be that the voltage which characterises the photoelectrolysis initiation increases when the photocurrent decreases.

In Fig. 2 it was represented the square photocurrent versus bias voltage in the same sample. From the bibliography [4] we know that this dependence must be a straight line, and its intersection with the abscisa gives the flat band potential value. Our dependence are not linear but it could be estimated the flat band potential to 0.45 V.

The behaviour of the $\alpha\text{-Fe}_2\text{O}_3$ impurified with 0.1% ZrO_2 and 0.1% TiO_2 respectively, is similar.

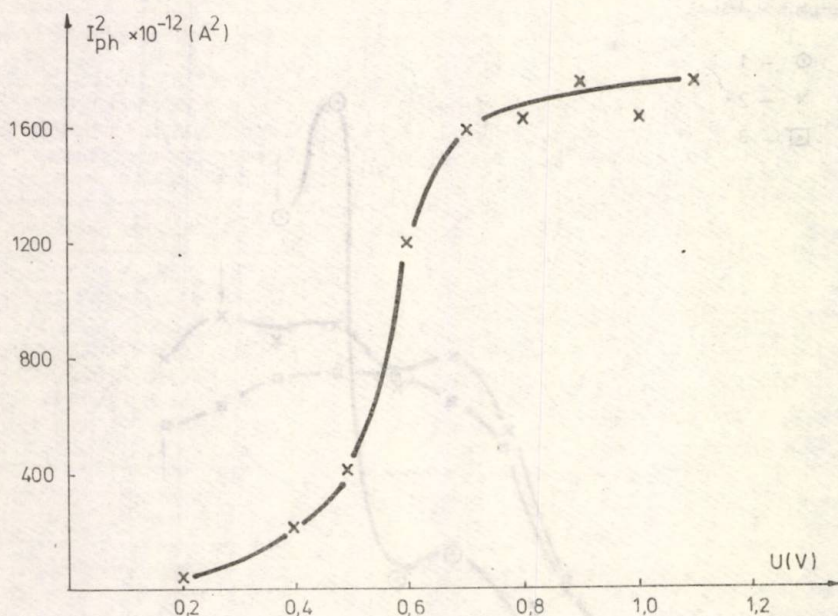


Fig. 2 — Square photocurrent versus potential in the sample impurified with 0.1% Nb_2O_5 .

4. Conclusions. The tested composition guide us to the conclusions that in the samples with low electrical resistance (impurified with 1%), the space charge depletion region is inefficient, and in the samples with relative large resistance, the photocurrent is small because of a too large voltage drop on the semiconductor electrode. We shall investigate the composition with impurification in the vicinity of 0.1%.

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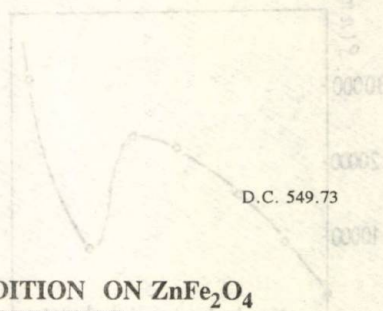
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FOTOANOZI DIN $\alpha-Fe_2O_3$ SINTERIZAT

(Rezumat)

Se studiază comportarea oxidului feric impurificat cu ZrO_2 , TiO_2 respectiv Nb_2O_5 , ca fotoanod semiconductor.



THE INFLUENCE OF CALCIUM ADDITION ON ZnFe_2O_4 FERRITE DIELECTRIC PROPERTIES

BY

DORIN CONDURACHE

1. Introduction. It is well known that the Ca^{2+} ions can enter the spinel lattice in a certain proportion [1]...[4], the relatively small solubility of Ca ions being determined by their relatively large ionic radius (0.94 Å) [5], as compared with the dimensions of the *A* tetrahedral and *B* octahedral interstices (0.297 Å and 0.546 Å respectively [6]), resulting in an alteration of the crystal lattice. The addition of small amounts of Ca in the ferrite strongly affects its dielectric and magnetic properties [2], [4] especially when the addition exceeds the stoichiometry, increasing the overall resistivity and reducing the dielectric losses.

This work is meant to establish the Ca ions amount limit added over the stoichiometry, that still does not imply the segregation of a secondary phase from the basis ferrite.

The influence of calcium oxide addition, exceeding the stoichiometry with 10% wt at most, in ZnFe_2O_4 ferrite, on its conductivity and dielectric properties is studied. An important dependence of these properties on the addition percentage was established. The results are discussed considering the Ca^{2+} ions partial solubility in the spinel lattice.

2. Experimental Procedure. The samples were prepared from oxides with purity exceeding 99.6% by the well known ceramic technology. After a 6 hours sintering at 1280°C the samples were cooled at a rate of about 2°C/min.

Seven samples were prepared characterized by CaO additions of 0.1%wt, 0.2%wt, 0.4%wt, 1%wt, 2%wt, 4%wt, 10%wt. The resistivity, permittivity and dielectric losses were determined by means of the Wheatstone bridge and double T admittance bridge. The dielectric properties were determined at a frequency of 1 kHz and all measurements were performed at room temperature.

3. Results and Discussion. In Fig. 1 the resistivity is given vs. CaO addition. One can see that the resistivity increases at the beginning, reaching a maximum value

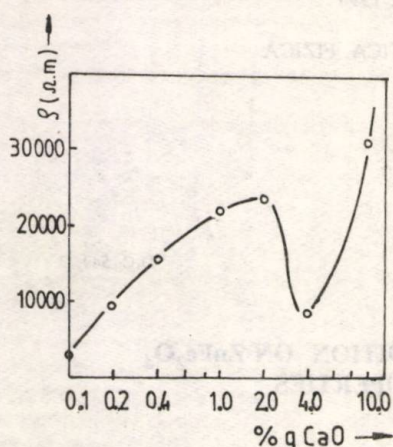
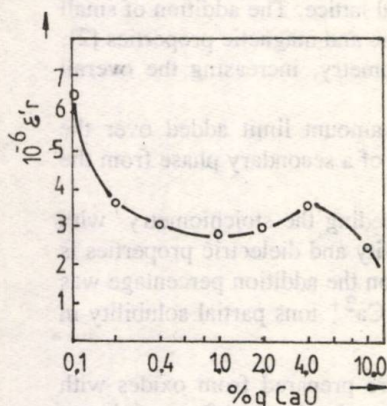


Fig. 1 — Resistivity vs. CaO addition.

of Ca ions within the spinel lattice and to the substitution of some Zn ions from the tetrahedral places resulting in the generation of monophasic quasispinel products, where vacancies appear due to the samples nonstoichiometric conditions.

Fig. 2 — Dielectric permittivity ϵ_r vs. Ca addition.

for 2% addition, then it decreases to a minimum value corresponding to a 4% CaO addition and finally goes quickly to very high values. The aspect of the curve is similar to that presented in [2] for NiFe_2O_4 samples with CaO addition, but both maximum and minimum are yet shifted to higher CaO amounts.

The explanation of the resistivity dependence on the Ca addition is essentially the same as given in [2], [7]...[9], namely that small Ca amount are migrating at crystallite limits making a highly resistive film with the thickness increasing with Ca addition. The resistivity decrease for 2...4% of CaO in ZnFe_2O_4 ferrite (for 0.2 to 0.5% in NiFe_2O_4 ferrite respectively [2]) corresponds to a massive penetration

The resistivity increase observed when CaO exceeds 4% is due to the appearance of Ca — Fe secondary phase.

The shift of the curve descending section for CaO higher amounts results from the higher Ca solubility in Zn ferrite, as compared to Ni ferrite. The Zn ionic radius being 0.83 Å it exceeds Ni ionic radius of 0.72 Å, being closer to Ca ionic radius [5].

In Fig. 2 the dependence of dielectric permittivity ϵ_r vs. Ca addition is given. A step decrease is seen for small additions followed by a slight increase for 4% CaO addition and another decrease for larger CaO amounts. The aspect of the curve is generally the mirror image of that given in Fig. 1.

The antisymmetrical variations of permittivity and resistivity respectively vs. composition is well known in the literature, for Ni ferrite impurified with Ca a proportionality being found between ϵ_r and $1/\rho$ [4]. It can be explained by the electronic exchange $\text{Fe}^{2+} \rightleftharpoons \text{Fe}^{3+}$ resulting in both the conduction process [10], [11] and the appearance of the interface polarization in the

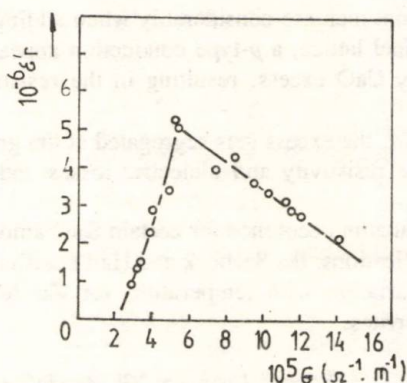


Fig. 3 — The dependence $\epsilon' = f(1/\rho)$.

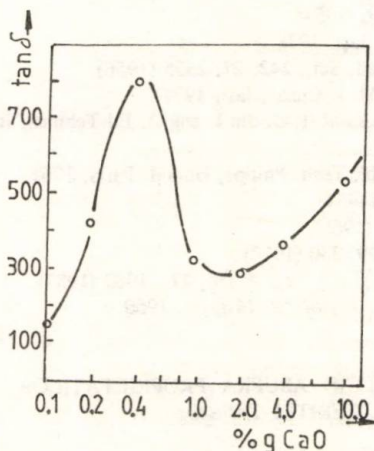


Fig. 4 — Dielectric loss variation vs. CaO addition.

ases.

In Fig. 4 the dielectric loss variation is given vs. CaO addition. The aspect of the curve is similar to that of the resistivity and contrary to that of the permittivity, $\tan \delta$ having a minimum for the composition at which ϵ' has a maximum. The same aspect was reported in [4] for NiFe_2O_4 ferrite impurified with CaO.

case of Ca amount segregated at the crystallites limits as a highly resistive film.

Trying a graphical representation in Fig. 3 similar to that considered in [4], i.e. $\epsilon' = f(1/\rho)$, we found a linear increase of permittivity at high values of resistivity, i.e. for samples where the Ca addition appears at crystallite limits as CaO (for additions up to 1...2%) or as CaFe compounds (for additions exceeding 4%).

For small amounts of Ca (less than 1%) and respectively for 4% addition corresponding to the maximum solubility the curve goes down suggesting rather a vacancies conduction than an electronic conduction.

Under this conduction the low mobility of the vacancies participating in the electric conduction determines a poor polarization characterized by a small dielectric constant explaining the descending section of the graphic. The vacancies occurrence in the ferrite at compositions for which Ca is solved into lattice may be explained as follows. For these compositions a number of vacancies appears playing the part of traps and catching some electrons that have participated in the $\text{Fe}^{2+} \rightleftharpoons \text{Fe}^{3+}$ exchange, the vacancies left after the capture playing the leading part in the conduction and polarization processes. When Ca segregation increases the vacancies concentration decreases and the concentration of free electrons increases too, while that of the vacancies decreases.

4. Conclusions. 1. Calcium enters partly in the spinel lattice. For small Ca

additions a slight composition heterogeneity appears at the grains level, Ca ions tending to migrate to the grains periphery.

2. The concentration gradient does not increase considerably when adding Ca within the limits of its solubility in the spinel lattice, a p -type conduction appearing when the number of vacancies induced by CaO excess, resulting in the resistivity decrease.

3. For CaO amounts exceeding 4%wt, the excess gets segregated at the grains boundaries resulting in the increase of the resistivity and dielectric losses and the decrease of permittivity.

4. The hypothesis of vacancies conduction occurrence for certain CaO amounts in ZnFe_2O_4 ferrite needs experimental verifications; the Seebeck and Hall coefficients are to be determined as well as their variation with temperature for Zn ferrite impurified with Ca, as well as for other ferrites.

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INFLUENȚA ADAOSULUI DE CALCIU ASUPRA PROPRIETĂȚILOR DIELECTRICE ALE FERITEI ZnFe_2O_4

(Rezumat)

Se studiază influența adaosului de Ca peste stoechiometrie, până la 10% în greutate, asupra conductivității și a proprietăților dielectrice ale feritei ZnFe_2O_4 .

Rezultatele sunt discutate în legătură cu limitele de solubilitate ale CaO în rețeaua spinelică.

RECENZII

BOOK REVIEWS

РЕЦЕНЗИИ

EARL PERRY, **Geometry: Axiomatic Developments with Problem Solving**. Marcel Dekker, Inc., New York, 1992, 354 pp., ISBN 0-8247-8727-7.

This interesting book written in a clear, formal style, provides a thorough presentation of the subject, furnishing precisely defined concepts, wellformed proofs, and carefully stated axioms and theorems.

Examining topics in their natural order and stressing the axiomatic development of several geometric systems, this highly organized text presents *Logic and Techniques Devoted to Developing Proofs. Finite Geometries of Young and Fano. The Properties of an Axiomatic System. An Extensive Geometry Based on Hilbert's Axioms. Classical Euclidean Geometry of Triangles and Circles. Cross Ratio and Harmonic Sets and Applications. Some Topics in Hyperbolic, Non-Euclidean Geometry. An Introduction to Transformation Geometry* and many other subjects.

Depending only on arithmetic, simple algebra, without reference to calculus, advanced algebra, linear algebra or trigonometry, **Geometry: Axiomatic Developments with Problem Solving** is an excellent primary text for all upper-level undergraduate and graduate one- and two-quarter courses in geometry, foundations of geometry, and geometry for teachers.

P. S. MILOJEVIC (Ed.), **Nonlinear Functional Analysis. Lecture Notes in Pure and Applied Mathematics**, 121, Marcel Dekker, Inc., New York — Basel, 1990, 264 pp., ISBN 0-8247-8255-0.

This volume consists of a collection of papers, most of which are based on the lectures presented at the *Special Session on Nonlinear Functional Analysis of the American Mathematical Society Regional Meeting*, held at New Jersey Institute of Technology in Newark, New Jersey, on 25-26 April, 1987.

The topics covered in the proceedings are *Global Invertibility and Finite Solvability on Nonlinear Differential Equations; Existence of Positive Eigenvalue and Eigenvalues for Compact and*

Condensing-like Maps; Recent Progress in Hamiltonian Dynamics and Symplectic Geometry; Solvability of Semilinear Operators and Hyperbolic Differential Equations; Operator-Valued Means and Their Iterates; Existence of Multiple Critical Points of Functionals Preserving an Order Structure.

The paper includes in this volume are: *Global Invertibility and Finite Solvability* (V. C a f a g n a); *Boundary Conditions and Vanishing Index for Contractions and Condensing Maps* (G. F o u r n i e r, M. M a r t l l i); *Recent Progress in Symplectic Geometry* (H. H o f e r); *Solvability of Semilinear Operator Equations and Applications to Semilinear Hyperbolic Equations* (P. S. M i l o j e v i c); *Some Remarks on Operator-Valued Means* (Roger D. N u s s b a u m); *Recent Progress in Periodic Orbits of Autonomous Hamiltonian Systems and Applications to Symplectic Geometry* (Claude V i t e r b o); *Critical Points in the Presence of Order Structures* (Krzysztof W y s o c k i).

Every paper contains many bibliographical references.

Romulus Ștefan

JEROME P. KEATING, ROBERT L. MASON and PRANAB K. SEN, **Pitman's Measure of Closeness: A Comparison of Statistical Estimators**. Society for Industrial and Applied Mathematics, SIAM, Philadelphia, 1993, 226 pp., ISBN 0-89871-308-0.

Pitman's measure of closeness is based on the probabilities of the relative closeness of competing estimators to an unknown parameter. For example a confidence interval is a measure of concentration (closeness) of an estimator about the true value of the parameter. PMC can be used to define a Pitman-closen estimator and a Pitman-closest estimator within a class D.

This book presents *A Unified Development of the Origins, Nature, Methods and Applications of Pitman's Measure of Closeness as a Criterion in Estimation*. PMC is based on the probabilities of the closeness of computing estimators to an unknown parameter.

The intended audience for this book consists of two groups: the first group includes statisticians and mathematicians who are research oriented and would like to know more about a new and growing area of statistics. It also includes those who have some formal training in the theory of statistics or interest in the estimation field. The second group for whom this book is intended includes graduate students in statistics courses in colleges and universities.

The topics selected for inclusion in this monography constitute a compilation of many research papers on PMC. It contains six chapters. Throughout the book a serious effort has been made to present both the merits and drawbacks of PMC.

Ch. 1, *Introduction*, begins with a *Philosophical Perspective*, presents the *Motivation for Exploring PMC* and the notation to be used throughout the book. Ch. 2, *Development of Pitman's Measure of Closeness*, contains *Discussions on the Development of PMC*, including its *History*, the *Concept of Risk*, the *Competition with MSE* and the *Role of the Loss Function*. Ch. 3, *Anomalies with PMC*, explores the *Operational Issue Involved in Adopting PMC as a Criterion*. Ch. 4, *Pairwise Comparisons*, establishes the *Fundamental Results of Pairwise Comparisons Based on PMC*. Ch. 5, *Pitman-Closest Estimators*, connects *Results of PMC* with *Wellaccepted Notions in Statistical Inference* such as robustness, equivariance and median unbiased estimation. Ch. 6, *Asymptotics and PMC*, contains results on asymptotics, including the *Optimality of BAN Estimators* according to the Pitman closeness criterion and general equivalence results of Pitman, Bayes and maximum likelihood estimators.

Romulus Ștefan

ILYA J. BAKELMAN (Ed.), *Geometric Analysis and Nonlinear Partial Differential Equations. Lecture Notes in Pure and Applied Mathematics*, 144, Marcel Dekker, Inc., New York — Basel, 1993, 308 pp., ISBN 0-8247-8897-4.

This book is the result of a special *Session at the 861st Meeting of the American Mathematical Society* held at the University of North Texas at Denton, Texas. The purpose of this Session was to present and develop *Research in Boundary Value Problems for Nonlinear Partial Differential Equations and Related Problems in the Theory of Convex Bodies, Differential Geometry and Mathematical Physics*.

The book contains four parts.

Part I, *Geometric Methods in Nonlinear Elliptic Partial Differential Equations and Applied Problems*, contains 8 works: *Geometric Inequalities and Estimates of Solutions for Nonlinear Euler-Lagrange Equation and Applied Problems* (Ilya J. Bakelman, William I. Perry); *Qualitative Behavior of Solutions to a System of Partial Differential Equations from Nonlinear Elasticity* (Patricia Bauman); *Asymptotic Approximations to the Fundamental Solutions of Differential Equations on Manifolds* (S.A. Fulling); *Harmonic Maps with Nontrivial Higher-Dimensional Singularities* (Guojun Liao, Nathan Smale); *Elliptic Systems for a Medium with Microstructure* (R. E. Showalter, N.J. Walkington); *Asymptotic Behavior of Positive Decreasing Solutions of $y'' = F(t, y, y')$* (Steven D. Taliaferro); *Uniqueness of Capillary Surfaces in Wedges and Cones* (Thomas I. Vogel); *A Neumann Evolution Problem for Plastic Antiplanar Shear* (Xiaodong Zhou).

Part II, *Convex Bodies and Related Topics*, contains 6 works: *Axiomatic Convex Potential Theory* (E.M.J. Bertin); *Area-Reducing Flows* (Xiaoxi Chen); *Double Normals Characterize Bodies of Constant Width in Riemannian Manifolds* (Boris V. Dekster); *Quasi-time Functions in Lorentzian Geometry* (Paul E. Ehrlich, Gerard G. Emch); *The Weyl Problem for Surfaces of Nonnegative Curvature* (Joseph A. Iai); *Singularities and Conformal Scalar Curvature* (Robert C. McOwen).

Part III includes two papers that touch upon the *Entire Subject Matter of this Book*. The first of them is devoted to *Recent Investigations of Monge-Ampère Equations*. The second paper discusses the *Generalization of the Isoperimetric Inequality to Two-dimensional Convex Surface Whose Integral Gaussian Curvature is Less than 2π* . Authors, Ilya Bakelman and Steven Taliaferro.

Part IV, *Problems: Open Problems in the Geometry of Equilibrium Configurations* (Robert Gulliver, Henry Wente).

Romulus Ștefan

HANS G. KAPER and MARC GARBEY (Eds.), *Asymptotic Analysis and Numerical Solution of Partial Differential Equations. Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, Inc., New York — Basel — Hong Kong, 1992, 271 pp., ISBN 0-8247-8538-X.

This volume contains the proceedings of the workshop *Asymptotic Analysis and Numerical Solution of Partial Differential Equations*, held at Argonne National Laboratory, February 26-28, 1990. Here over forty mathematicians, engineers and computer scientists met to hear six invited and more than a dozen contributed presentations. The volume provides the record of the workshop.

It contains six parts: the first part presents *Differential Methods of Singular Perturbation Theory* (uniform approximations of integrals, geometrical methods to solve two-point boundary value problems, the analyzing of strongly nonlinear oscillators with dissipative perturbations); the second part presents the *Asymptotic-Induced Numerical Methods Based on Domain Decomposition* (with examples for the convective diffusion equation or for the applied viscous and nonviscous conservation

laws); the third part gives *Examples of Results where the Asymptotic Analysis and Numerical Analysis Strongly Interact*; the fourth part concerns *Physical Problems with Critical Parameters and the Efficiency of Asymptotic Analysis* in such cases (with examples from oceanography, the inviscid steady transonic flows past a thin object, the nonlinear diffusion equations, etc.); the fifth part is devoted to the *Analysis of Strongly Nonlinear Problems*; and finally, the sixth part contains a single paper made by three of the organizers — Mark G a r b e y, Hans K a p e r and Man K a m K w o n g — concerning the *Connection Between the Symbolic Manipulation Software and the Study of Differential Equations*.

Gabriela Varvara

SHEPHERD BOYD, LAURENT EL GHAOU, ERIC FERON and VENKATARAMANAN BALAKRISHNAN, *Linear Matrix Inequalities in System and Control Theory*. Science and Industry Advance with Mathematics, SIAM, Philadelphia, 1994, XII + 193 pp., ISBN 0-89871-334-X.

In this book the authors reduce a *Wide Variety of Problems Arising in System and Control Theory to a Handful of Convex and Quasiconvex Optimization Problems* that involve linear matrix inequalities. These optimization problems can be solved using recently developed numerical algorithms that not only are polynomial-time but also work very well in practice; the reduction therefore can be considered a solution to the original problems. The book opens up an important new research area in which convex optimization is combined with system and control theory, resulting in the solution of a large number of previously unsolved problems.

C o n t e n t s. *Preface.* Ch. 1, *Introduction:* Overview; A Brief History of LMIs in Control Theory; Notes on the Style of the Book; Origin of the Book. Ch. 2, *Some Standard Problems Involving LMIs:* Linear Matrix Inequalities; Some Standard Problems; Ellipsoid Algorithm; Interior-Point Methods; Strict and Nonstrict LMIs; Miscellaneous Results on Matrix Inequalities; Some LMI Problems with Analytic Solutions. Ch. 3, *Some Matrix Problems:* Minimizing Condition Number by Scaling; Minimizing Condition Number of a Positive-Definite Matrix; Minimizing Norm by Scaling; Rescaling a Matrix Positive-Definite; Matrix Completion Problems; Quadratic Approximation of a Polytopic Norm; Ellipsoidal Approximation. Ch. 4, *Linear Differential Inclusions:* Differential Inclusions; Some Specific LDIs; Nonlinear System Analysis via LDIs. Ch. 5: *Analysis of LDIs; State Properties:* Quadratic Stability; Invariant Ellipsoids. Ch. 6, *Analysis of LDIs: Input/Output Properties:* Input-to-State Properties; State-to-Output Properties; Input-to-Output Properties. Ch. 7, *State-Feedback Synthesis for LDIs:* Static State-Feedback Controllers; State Properties; Input-to-State Properties; State-to-Output Properties; Input-to-Output Properties; Observer-Based Controllers for Nonlinear Systems. Ch. 8, *Lur'e and Multiplier Methods:* Analysis of Lur'e Systems; Integral Quadratic Constraints; Multiplier for Systems with Unknown Parameters. Ch. 9, *Systems with Multiplicative Noise:* Analysis of Systems with Multiplicative Noise; State-Feedback Synthesis. Ch. 10, *Miscellaneous Problems:* Optimization over an Affine Family of Linear Systems; Analysis of Systems with LTI Perturbations; Positive Orthant Stabilizability; Linear Systems with Delays; Interpolation Problems; The Inverse Problem of Optimal Control; Multi-Criterion LQG; Nonconvex Multi-Criterion Quadratic Problems; System Realization Problems. *Notation. List of Acronyms. Bibliography. Index.*

The book identifies a handful of standard optimization problems that are general (a wide variety of problems from system and control theory can be reduced to them) as well as specific (specialized numerical algorithms can be devised for them).

It catalogs a diverse list of problems in system and control theory that can be reduced to the standard optimization problems. Problems considered are analysis and state-feedback design for uncertain systems, matrix analysis problems, and many others.

K. D. ELWORTHY (Ed.), **Differential Equations, Dynamical Systems and Control Science**. Marcel Dekker, Inc., New York, 1994, 946 pp. ISBN 0-8247-8904-0.

Detailing the historical setting of modern control engineering, this remarkable reference — a Festschrift in honor of world-renowned mathematician Lawrence Markus — presents *Recent Developments in the Areas of Differential Equations, Dynamical Systems and Control of Finite and Infinite Dimensional Systems*.

Focusing on *Current Trends in Differential Equations and Dynamical Systems Research* — from parameter dependence of solutions to robust control laws for infinite dimensional systems — the book highlights the many contributions to the aerospace industry made by *Control Engineering Research*, describes *General Results of Stability, Exponential Stability and Lyapunov Functions* for relating structural properties of differential equations, discusses *Approximate Solutions for the Schrödinger and Navier-Stokes Equations*, considers various *Theoretical Questions for Control Systems with Infinite Dimensional State Spaces or Time Delay*, relates *Stabilizing Solutions for Riccati Inequalities to Disturbance Attenuation*, covers *Robust Stability, Nonlinear Models and Control of Two-dimensional Systems*, examines *Techniques for Numerical Solutions and Approximations of Nonlinear Infinite Dimensional System Models* and more other interesting topics.

MURRAY S. KLAMKIN (Ed.), **Mathematical Modelling: Classroom Notes in Applied Mathematics**. Society for Industrial and Applied Mathematics SIAM, Philadelphia, 1987, XIV + 338 pp., ISBN 0-89871-204-1.

Designed for classroom use, this book contains *Short, Self-contained Mathematical Models of Problems in the Physical, Mathematical and Biological Sciences*, first published in the *Classroom Notes* section of the SIAM Review from 1975-1985.

The problems provide an ideal way to make complex subject matter more accessible to the student through the use of concrete applications. Each section has extensive supplementary references provided by the editor from his years of experience with mathematical modelling.

C o n t e n t s. *Introduction. Part I, Physical and Mathematical Sciences: Mechanics, Heat Transfer, Traffic Flow, Electrical Networks, Communication and Coding, Graph Theory and Applications, Applied Geometry, Numerical Analysis, Mathematical Economics, Optimization, Music, Stochastic Models. Part II, Life Sciences: Population Models, Medical Models.*

INGRID DAUBECHIES, **Ten Lectures on Wavelets**. Science and Industry Advance with Mathematics, SIAM, Philadelphia, 1992, XIX + 357 pp., ISBN 0-89871-274-2.

Wavelets are a mathematical development that may revolutionize the world of information storage and retrieval according to many experts. *Wavelets Are a Fairly Simple Mathematical Tool* now being *Applied to the Compression of Data* — such as fingerprints, weather satellite photographs and medical X-rays — *that Were Previously thought to be Impossible to Condense Without Losing Crucial Details*.

This monograph contains 10 lectures presented by Dr. Ingrid Daubechies as the principal speaker at the **1990 CMBS-NSF Conference on Wavelets and Applications**. The author is a member of the technical staff at AT&T Bell Laboratories while on leave from her tenured position in the Theoretical Physics Department at the Free University, Brussels. She is a full professor in the Mathematics Department at Rutgers University, is a frequent lecturer, has published 46 papers on several aspects of the wavelet transform and has developed a collection of wavelets that are remarkably efficient.

C o n t e n t s. *Introduction; Preliminaries and Notation; The What, and How of Wavelets;*

The Continuous Wavelet Transform; Discrete Wavelet Transforms: Frames; Time-Frequency Density and Orthonormal Bases; Orthonormal Bases of Wavelets and Multiresolutional Analysis; Orthonormal Bases of Compactly Supported Wavelets; More About the Regularity of Compactly Supported Wavelets; Symmetry for Compactly Supported Wavelet Bases; Characterization of Functional Spaces by Means of Wavelets; Generalizations and Tricks for Orthonormal Wavelet Bases; Reference; Indexes.

Mathematicians or other scientists are engineers interested in the applications (in signal analysis, time-frequency methods, numerical analysis, etc.) of wavelets will benefit most from this book. A background in Fourier analysis and some real analysis is suggested. This volume is also appropriate for graduate or advanced undergraduate courses on wavelets.

JOHN R. HOOK and HENRI EDGAR HALL, *Solid State Physics*. John Wiley & Sons, Chichester, New York, 1991, XXII + 474 pp.

This is a very good introduction to some *Advanced Topics in Solid State Physics*, especially *Recent Developments of Semiconductor Physics and Devices*. It is clear, concise and complete and provides the best available treatment of many if not most of the topics it treats. In particular, it is more accessible than Kittel *Introduction to Solid State Physics*, to which is often refers (and which it is not attempting to replace).

One could in principle use this book as a first course in solid state physics, as the opening chapters provide a *Complete Physical Treatment of the Structure and Dynamics of Crystal*, up to and including the Schrödinger equation. However, it would be better to use the first two chapters as the review it is intended to be, which makes the book more suited for a second course in solid state. The third chapter provides an excellent *Introduction to the Free Electron Model*. The other chapters are: 4, *The Effect of the Periodic Lattice Potential Energy Bands*. 5, *Semiconductors*. 6, *Semiconductor Devices*. 7, *Diamagnetism and Paramagnetism*. 8, *Magnetic Order*. 9, *Electric Properties of Insulators*. 10, *Superconductivity*. 11, *Waves in Crystals*. 12, *Scattering of Neutrons and Electrons from Solids*. 13, *Real Metals*. 14, *Low-dimensional Systems*.

The presentation of the material is the most pedagogical currently available and provides a nice complement to the treatment in Kittel. There are four appendices discussing: *Coupled Probability Amplitudes; Electric and Magnetic Fields Inside Materials; Quantum Mechanics of an Electron in a Magnetic Field; The Exchange Energy*. There are also a fair number of problems, many of which are (deliberately) quite challenging.

In sum, I like the book very much, primarily for its treatment of the real metals and low dimensional systems. I am very glad to have this fine book in my library.

Corneliu Ciubotariu





